REPRESENTATION THEORY OF THE NONSTANDARD HECKE ALGEBRA

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ABSTRACT. The nonstandard Hecke algebra $\check{\mathscr{H}}_r$ was defined by Mulmuley and Sohoni to study the Kronecker problem. We study a quotient $\check{\mathscr{H}}_{r,2}$ of $\check{\mathscr{H}}_r$, called the nonstandard Temperley-Lieb algebra, which is a subalgebra of the symmetric square of the Temperley-Lieb algebra TL_r. We give a complete description of its irreducible representations. We find that the restriction of an $\check{\mathscr{H}}_{r,2}$-irreducible to $\check{\mathscr{H}}_{r-1,2}$ is multiplicity-free, and as a consequence, any $\check{\mathscr{H}}_{r,2}$-irreducible has a seminormal basis that is unique up to a diagonal transformation.

1. Introduction

Let $\mathscr{H}_r$ be the type A_{r-1} Hecke algebra over $\mathbf{A} = \mathbb{Z}[u, u^{-1}]$ and set $K := \mathbb{Q}(u)$. The nonstandard Hecke algebra $\check{\mathscr{H}}_r$ is the subalgebra of $\mathscr{H}_r \otimes \mathscr{H}_r$ generated by

$$\mathcal{P}_i := C'_{s_i} \otimes C'_{s_i} + C_{s_i} \otimes C_{s_i},\ i \in [r-1],$$

where C'_{s_i} and C_{s_i} are the simplest lower and upper Kazhdan-Lusztig basis elements, which are proportional to the trivial and sign idempotents of the parabolic sub-Hecke algebra $K(\mathscr{H}_r)_{\{s_i\}}$. The nonstandard Hecke algebra was introduced by Mulmuley and Sohoni in [11] to study the Kronecker problem. The hope was that the inclusion $\check{\Delta} : \check{\mathscr{H}}_r \to \mathscr{H}_r \otimes \mathscr{H}_r$ would quantize the coproduct $\Delta : \mathbb{Z}\mathcal{S}_r \to \mathbb{Z}\mathcal{S}_r \otimes \mathbb{Z}\mathcal{S}_r$ of the group algebra $\mathbb{Z}\mathcal{S}_r$ and canonical basis theory could be applied to obtain formulas for Kronecker coefficients. Unfortunately, this does not work in a straightforward way since the algebra $\check{\mathscr{H}}_r$ is almost as big as $\mathscr{H}_r \otimes \mathscr{H}_r$ and has $\mathbf{A}$-rank much larger than $r!$, even though $\check{\Delta}$ is in a certain sense the quantization of Δ with image as small as possible (see [2, Remark 11.4]). Nonetheless, as discussed in [2, §1], [10, 9], and briefly in this paper, the nonstandard Hecke algebra may still be useful for the Kronecker problem.

Though the nonstandard Hecke algebra has yet to prove its importance for the Kronecker problem, it is an interesting problem in its own right to determine all the irreducible representations of $K\check{\mathscr{H}}_r$. This problem is difficult, but within reach. In this paper, we solve an easier version of this problem.

It is shown in [2] that $K\check{\mathscr{H}}_r$ is semisimple. Let τ be the flip involution of $\mathscr{H}_r \otimes \mathscr{H}_r$ given by $h_1 \otimes h_2 \mapsto h_2 \otimes h_1$ and let $\theta : \mathscr{H}_r \to \mathscr{H}_r$ be the $\mathbf{A}$-algebra involution defined by $\theta(T_{s_i}) = -T_{s_i}^{-1},\ i \in [r-1]$. Twisting an $\mathscr{H}_r$-irreducible by θ corresponds to transposing its shape. The algebra $\check{\mathscr{H}}_r$ is a subalgebra of $(S^2\mathscr{H}_r)^{\theta\otimes\theta}$, the subalgebra of $\mathscr{H}_r \otimes \mathscr{H}_r$

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fixed by $\theta \otimes \theta$ and τ. Based on computations for $r \leq 6$, it appears that most of the $K\check{\mathscr{H}}_r$-irreducibles are restrictions of $K(S^2\mathscr{H}_r)^{\theta\otimes\theta}$-irreducibles, except for the trivial and sign representations of $K\check{\mathscr{H}}_r$.

In this paper we focus on the simpler problem of determining the irreducibles of the nonstandard Temperley-Lieb algebra $\check{\mathscr{H}}_{r,2}$, which is a quotient of $\check{\mathscr{H}}_r$. The algebra $\check{\mathscr{H}}_{r,2}$ is the subalgebra of $\mathscr{H}_{r,2} \otimes \mathscr{H}_{r,2}$ generated by $\mathcal{P}_i := C'_{s_i} \otimes C'_{s_i} + C_{s_i} \otimes C_{s_i},\ i \in [r-1]$, where $\mathscr{H}_{r,2}$ is the Temperley-Lieb algebra (see §5).

The main result of this paper is a complete description of the $K\check{\mathscr{H}}_{r,2}$-irreducibles (Theorem 5.1). There are no surprises here: it is fairly easy to show that $K(\mathscr{H}_{r,2} \otimes \mathscr{H}_{r,2})$-irreducibles decompose into certain $K\check{\mathscr{H}}_{r,2}$-modules. The difficulty is showing that these modules are actually irreducible. We prove this by induction on r and by computing the action of $\mathcal{P}_{r-1}$ on these modules in terms of canonical bases. To carry out these computations, we use results from [5] about projecting the upper and lower canonical bases of a $K\mathscr{H}_r$-irreducible M_λ onto its $K\mathscr{H}_{r-1}$-irreducible isotypic components. We also use the well-known fact that the edge weight $\mu(x, w)$, $x, w \in \mathcal{S}_r$, of the $\mathcal{S}_r$-graph $\Gamma_{\mathcal{S}_r}$ is equal to 1 whenever x and w differ by a dual Knuth transformation (see §3.2 and §3.5).

One consequence of Theorem 5.1 is that the restriction of a $K\check{\mathscr{H}}_{r,2}$-irreducible to $K\check{\mathscr{H}}_{r-1,2}$ is multiplicity-free. Thus each $K\check{\mathscr{H}}_{r,2}$-irreducible has a seminormal basis (in the sense of [12]—see Definition 6.1) that is unique up to a diagonal transformation. This can also be used to define a seminormal basis for any $K(\mathscr{H}_{r,2} \otimes \mathscr{H}_{r,2})$-irreducible. Even though the irreducibles of $K\check{\mathscr{H}}_{r,2}$ are close to those of $K(\mathscr{H}_{r,2} \otimes \mathscr{H}_{r,2})$, the nonstandard Temperley-Lieb algebra offers something new: the seminormal basis of $M_\lambda \otimes M_\mu$ using the chain $K\check{\mathscr{H}}_{J_1} \subseteq \cdots \subseteq K\check{\mathscr{H}}_{J_{r-1}} \subseteq K\check{\mathscr{H}}_{J_r}$ is significantly different from the seminormal basis using the chain $K(\mathscr{H}_{1,2} \otimes \mathscr{H}_{1,2}) \subseteq \cdots \subseteq K(\mathscr{H}_{r-1,2} \otimes \mathscr{H}_{r-1,2}) \subseteq K(\mathscr{H}_{r,2} \otimes \mathscr{H}_{r,2})$, where $\check{\mathscr{H}}_{J_k}$ is the subalgebra of $\check{\mathscr{H}}_{r,2}$ generated by $\mathcal{P}_1, \ldots, \mathcal{P}_{k-1}$.

We are interested in these seminormal bases primarily as a tool for constructing a canonical basis of a $K\check{\mathscr{H}}_{r,2}$-irreducible that is compatible with its decomposition into irreducibles at $u = 1$, as described in [2, §19]. Thus even though the representation theory of the nonstandard Hecke algebra alone is not enough to understand Kronecker coefficients, there is hope that the seminormal bases will yield a better understanding of Kronecker coefficients. In fact, [9] gives a conjectural scheme for constructing a canonical basis of a $K\check{\mathscr{H}}_{r,2}$-irreducible using its seminormal basis, but this remains conjectural and we do not know how to use it to understand Kronecker coefficients.

This paper is organized as follows: sections 2–4 are preparatory: §3 reviews the necessary facts about canonical bases of $\mathscr{H}_r$ and their behavior under projection onto $K\check{\mathscr{H}}_{r-1}$-irreducibles; §4 gives some basic results about the representation theory of $\check{\mathscr{H}}_r$. Section 5 contains the statement and proof of the main theorem. Then in §6, seminormal bases of $K\check{\mathscr{H}}_{r,2}$-irreducibles are defined, and in §7, the dimension of $K\check{\mathscr{H}}_{r,2}$ is determined.

2. Partitions and tableaux

A *partition* λ of r of length $\ell(\lambda) = l$ is a sequence $(\lambda_1, \ldots, \lambda_l)$ such that $\lambda_1 \geq \cdots \geq \lambda_l > 0$ and $r = \sum_{i=1}^{l} \lambda_i$. The notation $\lambda \vdash r$ means that λ is a partition of r. Let $\mathscr{P}_r$ denote

the set of partitions of size r and $\mathscr{P}'_r$ the subset of $\mathscr{P}_r$ consisting of those partitions that are not a single row or column shape. The symbols $\trianglerighteq, \triangleright$ will denote dominance order on partitions. The conjugate partition λ' of a partition λ is the partition whose diagram is the transpose of the diagram of λ.

The set of standard Young tableaux is denoted SYT and the subset of SYT of shape λ is denoted SYT(λ). Tableaux are drawn in English notation, so that entries increase from north to south along columns and increase from west to east along rows. For a tableau T, $\text{sh}(T)$ denotes the shape of T.

For a word $\mathbf{k} = k_1 k_2 \dots k_r$, $k_i \in \mathbb{Z}_{>0}$, let $P(\mathbf{k}), Q(\mathbf{k})$ denote the insertion and recording tableaux produced by the Robinson-Schensted-Knuth (RSK) algorithm applied to $\mathbf{k}$. The notation Q^T denotes the transpose of an SYT Q, so that $\text{sh}(Q^T) = \text{sh}(Q)'$.

Let T be a tableau of shape λ. If b is a square of the diagram of λ, then T_b denotes the entry of T in the square b. If $\nu \subseteq \lambda$, then T_ν denotes the subtableau of T obtained by restricting T to the diagram of ν.

Let λ and μ be partitions of r. Throughout this paper, $a_1, \dots, a_{k_\lambda}$ (resp. $b_1, \dots, b_{k_\mu}$) will denote the outer corners of the diagram of λ (resp. μ) labeled so that a_{i+1} lies to the east of a_i (resp. b_{i+1} lies to the east of b_i), as in the following example.

λ μ (1)

3. Canonical bases of the Hecke algebra $\mathscr{H}_r$

Here we recall the definition of the Kazhdan-Lusztig basis elements C_w and C'_w and review the connection between cells in type A and tableaux combinatorics, following [5]. We then discuss dual equivalence graphs and recall some results of [5] about projecting canonical bases, which will make these bases fairly easy to work with in the proof of Theorem 5.1.

We work over the ground rings $\mathbf{A} = \mathbb{Z}[u, u^{-1}]$ and $K = \mathbb{Q}(u)$. Define K_0 (resp. K_∞) to be the subring of K consisting of rational functions with no pole at $u = 0$ (resp. $u = \infty$).

Let $\bar{\cdot}$ be the involution of K determined by $\overline{u} = u^{-1}$; it restricts to an involution of $\mathbf{A}$. For a nonnegative integer k, the $\bar{\cdot}$-invariant quantum integer is $[k] := \frac{u^k - u^{-k}}{u - u^{-1}} \in \mathbf{A}$. We also use the notation $[k]$ to denote the set $\{1, \dots, k\}$, but these usages should be easy to distinguish from context.

3.1. The Hecke algebra $\mathscr{H}(W)$.

Let (W, S) be a Coxeter group with length function ℓ and Bruhat order $<$. If $\ell(vw) = \ell(v) + \ell(w)$, then $vw = v \cdot w$ is a *reduced factorization.* The *right descent set* of $w \in W$ is $R(w) = \{s \in S : ws < w\}$.

For any $L \subseteq S$, the *parabolic subgroup* W_L is the subgroup of W generated by L.

The *Hecke algebra* $\mathscr{H}(W)$ of (W,S) is the free $\mathbf{A}$-module with standard basis $\{T_w : w \in W\}$ and relations generated by

$$\begin{array}{ll} T_vT_w = T_{vw} & \text{if } vw = v\cdot w \text{ is a reduced factorization,} \\ (T_s - u)(T_s + u^{-1}) = 0 & \text{if } s \in S. \end{array} \tag{2}$$

3.2. **The upper and lower canonical basis of $\mathscr{H}(W)$.** The *bar-involution*, $\bar{\cdot}$, of $\mathscr{H}(W)$ is the additive map from $\mathscr{H}(W)$ to itself extending the $\bar{\cdot}$-involution of $\mathbf{A}$ and satisfying $\overline{T_w} = T_{w^{-1}}^{-1}$. Observe that $\overline{T_s} = T_s^{-1} = T_s + u^{-1} - u$ for $s \in S$. Some simple $\bar{\cdot}$-invariant elements of $\mathscr{H}(W)$ are $C'_{\mathrm{id}} := T_{\mathrm{id}}$, $C_s := T_s - u = T_s^{-1} - u^{-1}$, and $C'_s := T_s + u^{-1} = T_s^{-1} + u$, $s \in S$.

Define the lattices $(\mathscr{H}_r)_{\mathbb{Z}[u]} := \mathbb{Z}[u]\{T_w : w \in W\}$ and $(\mathscr{H}_r)_{\mathbb{Z}[u^{-1}]} := \mathbb{Z}[u^{-1}]\{T_w : w \in W\}$ of $\mathscr{H}_r$. It is shown in [7] that

for each $w \in W$, there is a unique element $C_w \in \mathscr{H}(W)$ such that $\overline{C_w} = C_w$ and C_w is congruent to $T_w \mod u(\mathscr{H}_r)_{\mathbb{Z}[u]}$. (3)

The $\mathbf{A}$-basis $\Gamma_W := \{C_w : w \in W\}$ is the *upper canonical basis* of $\mathscr{H}(W)$ (we use this language to be consistent with that for crystal bases). Similarly,

for each $w \in W$, there is a unique element $C'_w \in \mathscr{H}(W)$ such that $\overline{C'_w} = C'_w$ and C'_w is congruent to $T_w \mod u^{-1}(\mathscr{H}_r)_{\mathbb{Z}[u^{-1}]}$. (4)

The $\mathbf{A}$-basis $\Gamma'_W := \{C'_w : w \in W\}$ is the *lower canonical basis* of $\mathscr{H}(W)$.

The coefficients of the lower canonical basis in terms of the standard basis are the *Kazhdan-Lusztig polynomials* $P'_{x,w}$:

$$C'_w = \sum_{x\in W} P'_{x,w} T_x. \tag{5}$$

(Our $P'_{x,w}$ are equal to $q^{(\ell(x)-\ell(w))/2}P_{x,w}$, where $P_{x,w}$ are the polynomials defined in [7] and $q^{1/2} = u$.) Now let $\mu(x,w) \in \mathbb{Z}$ be the coefficient of u^{-1} in $P'_{x,w}$ (resp. $P'_{w,x}$) if $x \le w$ (resp. $w \le x$). Then the right regular representation in terms of the canonical bases of $\mathscr{H}_r$ takes the following simple forms:

$$C'_wC'_s = \begin{cases} [2]C'_w & \text{if } s \in R(w), \\ \displaystyle\sum_{\{w'\in W : s\in R(w')\}} \mu(w',w)C'_{w'} & \text{if } s \notin R(w). \end{cases} \tag{6}$$

$$C_wC_s = \begin{cases} -[2]C_w & \text{if } s \in R(w), \\ \displaystyle\sum_{\{w'\in W : s\in R(w')\}} \mu(w',w)C_{w'} & \text{if } s \notin R(w). \end{cases} \tag{7}$$

The simplicity and sparsity of this action along with the fact that the right cells of Γ_W and Γ'_W often give rise to $\mathbb{C}(u)\otimes_{\mathbf{A}}\mathscr{H}(W)$-irreducibles are among the most amazing and useful properties of canonical bases.

We will make use of the following positivity result due to Kazhdan-Lusztig and Beilinson-Bernstein-Deligne-Gabber (see, for instance, [8]).

Theorem 3.1. *If (W,S) is crystallographic, then the integers $\mu(x,w)$ are nonnegative.*

3.3. **Cells.** We define cells in the general setting of modules with basis, as in [5] (this is similar to the notion of cells of Coxeter groups from [7]).

Let H be an R-algebra for some commutative ring R. Let M be a right H-module and Γ an R-basis of M. The preorder $\leq_\Gamma$ (also denoted $\leq_M$) on the vertex set Γ is generated by the relations

$$\delta \preceq_\Gamma \gamma \quad \begin{array}{c}\text{if there is an } h \in H \text{ such that } \delta \text{ appears with nonzero} \\ \text{coefficient in the expansion of } \gamma h \text{ in the basis } \Gamma.\end{array} \tag{8}$$

Equivalence classes of $\leq_\Gamma$ are the *right cells* of (M, Γ). The preorder $\leq_M$ induces a partial order on the right cells of M, which is also denoted $\leq_M$. We say that the right cells Λ and Λ' are isomorphic if $(R\Lambda, \Lambda)$ and $(R\Lambda', \Lambda')$ are isomorphic as modules with basis. Sometimes we speak of the right cells of M or right cells of Γ if the pair (M, Γ) is clear from context. We also use the terminology *right H-cells* when we want to make it clear that the algebra H is acting.

3.4. **Cells and tableaux.** Let $\mathscr{H}_r = \mathscr{H}(\mathcal{S}_r)$ be the type A Hecke algebra. For the remainder of the paper, set $S := \{s_1, \ldots, s_{r-1}\}$ and $J := \{s_1, \ldots, s_{r-2}\}$.

It is well known that $K\mathscr{H}_r := K \otimes_{\mathbf{A}} \mathscr{H}_r$ is semisimple and its irreducibles in bijection with partitions of r; let M_λ and $M_\lambda^{\mathbf{A}}$ be the $K\mathscr{H}_r$-irreducible and Specht module of $\mathscr{H}_r$ of shape $\lambda \vdash r$ (hence $M_\lambda \cong K \otimes_{\mathbf{A}} M_\lambda^{\mathbf{A}}$). For any $K\mathscr{H}_r$-module N and partition λ of r, let $p_{M_\lambda} : N \to N$ be the $K\mathscr{H}_r$-module projector with image the M_λ-isotypic component of N.

The work of Kazhdan and Lusztig [7] shows that the decomposition of $\Gamma_{\mathcal{S}_r}$ into right cells is $\Gamma_{\mathcal{S}_r} = \bigsqcup_{\lambda \vdash r,\, P \in \mathrm{SYT}(\lambda)} \Gamma_P$, where $\Gamma_P := \{C_w : P(w) = P\}$. Moreover, the right cells $\{\Gamma_P : \mathrm{sh}(P) = \lambda\}$ are all isomorphic, and, denoting any of these cells by Γ_λ, $\mathbf{A}\Gamma_\lambda \cong M_\lambda^{\mathbf{A}}$. Similarly, the decomposition of $\Gamma'_{\mathcal{S}_r}$ into right cells is $\Gamma'_{\mathcal{S}_r} = \bigsqcup_{\lambda \vdash r,\, P \in \mathrm{SYT}(\lambda)} \Gamma'_P$, where $\Gamma'_P := \{C'_w : P(w)^T = P\}$. Moreover, the right cells $\{\Gamma'_P : \mathrm{sh}(P) = \lambda\}$ are all isomorphic, and, denoting any of these cells by Γ'_λ, $\mathbf{A}\Gamma'_\lambda \cong M_\lambda^{\mathbf{A}}$. A combinatorial discussion of left cells in type A is given in [3, §4].

We refer to the basis Γ_λ of $M_\lambda^{\mathbf{A}}$ as the *upper canonical basis of* M_λ and denote it by $\{C_Q : Q \in \mathrm{SYT}(\lambda)\}$, where C_Q corresponds to C_w for any (every) $w \in \mathcal{S}_r$ with recording tableau Q. Similarly, the basis Γ'_λ of $M_\lambda^{\mathbf{A}}$ is the *lower canonical basis of* M_λ, denoted $\{C'_Q : Q \in \mathrm{SYT}(\lambda)\}$, where C'_Q corresponds to C'_w for any (every) $w \in \mathcal{S}_r$ with recording tableau Q^T. Note that with these labels the action of C_s on the upper canonical basis of M_λ is similar to (7), with $\mu(Q', Q) := \mu(w', w)$ for any w', w such that $P(w') = P(w)$, $Q' = Q(w'), Q = Q(w)$, and right descent sets

$$R(C_Q) = \{s_i : i+1 \text{ is strictly to the south of } i \text{ in } Q\}. \tag{9}$$

Similarly, the action of C'_s on $\{C'_Q : Q \in \mathrm{SYT}(\lambda)\}$ is similar to (6), with $\mu(Q', Q) := \mu(w', w)$ for any w', w such that $P(w')^T = P(w)^T$, $Q' = Q(w')^T, Q = Q(w)^T$, and right descent sets

$$R(C'_Q) = \{s_i : i+1 \text{ is strictly to the east of } i \text{ in } Q\}. \tag{10}$$

See Figure 1 for a picture of $\Gamma_{(3,2)}$ and $\Gamma'_{(3,2)}$ and right descent sets.

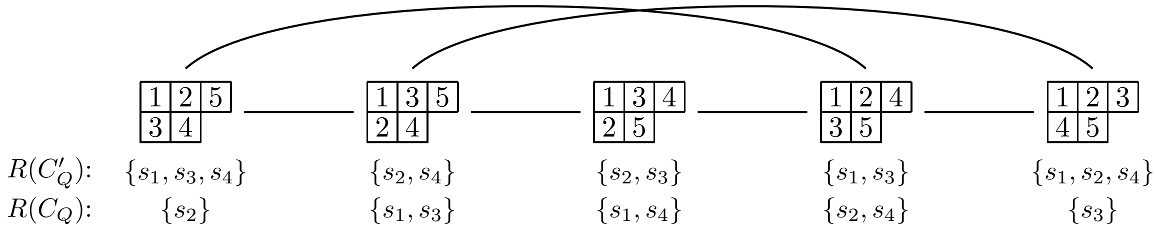

Figure 1: The $\mathcal{S}_r$-graph on Γ'_λ and Γ_λ. The presence (resp. absence) of an edge means that $\mu(Q', Q) = \mu(Q, Q')$ is 1 (resp. 0).

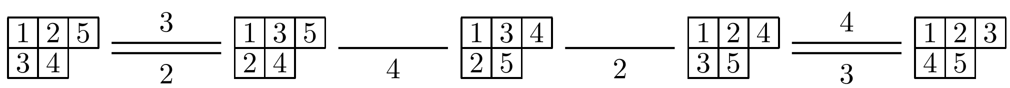

Figure 2: The DE graph on SYT((3, 2)).

3.5. **Dual equivalence graphs.** To work with canonical bases in the proof of Theorem 5.1, we make use the notion of dual equivalence graphs[1] from [1]. Given $T, T' \in \mathrm{SYT}(\lambda)$, we say that T and T' are related by a *dual Knuth transformation* at i if

(1) $|R(C'_T) \cap \{s_{i-1}, s_i\}| = |R(C'_{T'}) \cap \{s_{i-1}, s_i\}| = 1$,
(2) T' is obtained from T by swapping the entries i and $i+1$ in T or by swapping the entries $i-1$ and i in T.

If T and T' are related by a dual Knuth transformation at i, then we also say that there is a *DKT$_i$-edge* between T and T' and write $T \underset{i}{\leftrightsquigarrow} T'$. We write $T \leftrightsquigarrow T'$ if $T \underset{i}{\leftrightsquigarrow} T'$ for some i, $2 \leq i \leq r-1$.

Define the *dual equivalence graph (DE graph)* on $\mathrm{SYT}(\lambda)$ to be the graph with vertex set $\mathrm{SYT}(\lambda)$ and edges given by the DKT$_i$-edges for all i, $2 \leq i \leq r-1$.

We will freely use the result from [7] that $T \leftrightsquigarrow T'$ implies $\mu(T', T) = \mu(T, T') = 1$. Note that this means that the $\mathcal{S}_r$-graph on Γ'_λ (or Γ_λ) contains the underlying simple graph of the DE graph on $\mathrm{SYT}(\lambda)$ (compare Figures 1 and 2).

It is easy to see that (with the help of Figure 3)

for any distinct $i, j \in [k_\lambda]$, there exists at least one edge $T \underset{r-1}{\leftrightsquigarrow} T'$ in the DE graph on $\mathrm{SYT}(\lambda)$ with $T_{a_i} = r = T'_{a_j}$ and $T_{a_j} = r-1 = T'_{a_i}$. (11)

Here T_{a_i} denotes the entry of T in the square a_i (see §2).

[1]We use a slightly simplified version of the dual equivalence graphs from [1].

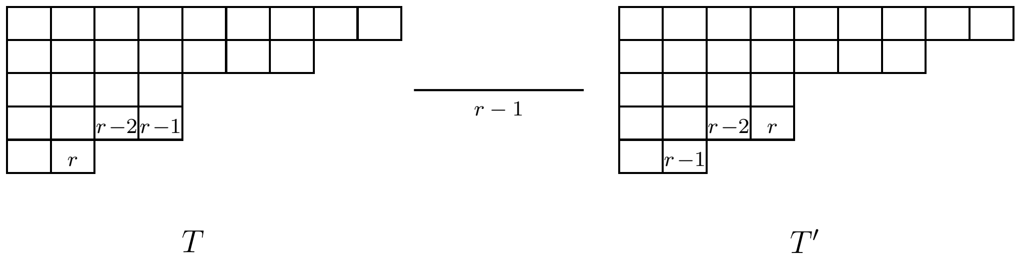

Figure 3: An edge of the DE graph on $\mathrm{SYT}(\lambda)$ as in (11) for $i = 1, j = 2$.

3.6. **Projected canonical bases.** Here we recall some results from [5] about projecting the upper and lower canonical bases of M_λ onto the $K\mathscr{H}_{r-1}$-irreducible isotypic components of M_λ. These results will make it fairly easy to work with these bases in the proof of Theorem 5.1.

For any $L \subseteq S$, define $(\tilde{C}_Q)^L$ to be the projection of C_Q onto the irreducible $K\mathscr{H}_L$-module corresponding to the right cell of $\mathrm{Res}_{K\mathscr{H}_L} K\Gamma_\lambda$ containing C_Q, where $\mathscr{H}_L$ denotes the parabolic sub-Hecke algebra of $\mathscr{H}_r$ with $\mathbf{A}$-basis $\{T_w : w \in (S_r)_L\}$. Define $(\tilde{C}'_Q)^L$ similarly. If $L = J := \{s_1, \ldots, s_{r-2}\}$, then by [3, §4], $(\tilde{C}_Q)^J$ (resp. $(\tilde{C}'_Q)^J$) is equal to $p_{M_\mu}(C_Q)$ (resp. $p_{M_\mu}(C'_Q)$), where $\mu = \mathrm{sh}(Q|_{[r-1]})$ and p_{M_μ} is defined in §3.4. Here, for a tableau Q and set $Z \subseteq \mathbb{Z}$, $Q|_Z$ denotes the subtableau of Q obtained by removing the entries not in Z.

Maintain the notation of (1) for the outer corners of λ. Define a partial order $\triangleleft_r$ on $\mathrm{SYT}(\lambda)$ by declaring $Q' \triangleleft_r Q$ whenever $\mathrm{sh}(Q'|_{[r-1]}) \triangleright \mathrm{sh}(Q|_{[r-1]})$. Recall that K_0 (resp. K_∞) is the subring of K consisting of rational functions with no pole at $u = 0$ (resp. $u = \infty$).

Lemma 3.2 ([5][2])**.** *The transition matrix expressing the projected basis $\{(\tilde{C}_Q)^J : Q \in SYT(\lambda)\}$ in terms of the upper canonical basis of M_λ is lower-unitriangular and is the identity at $u = 0$ and $u = \infty$ (i.e. $(\tilde{C}_Q)^J = C_Q + \sum_{Q' \triangleright_r Q} m_{Q'Q} C_{Q'}$, $m_{Q'Q} \in uK_0 \cap u^{-1}K_\infty$). The transition matrix expressing the projected basis $\{(\tilde{C}'_Q)^J : Q \in SYT(\lambda)\}$ in terms of the lower canonical basis of M_λ satisfies the same properties except is upper-unitriangular instead of lower-unitriangular (i.e. $(\tilde{C}'_Q)^J = C'_Q + \sum_{Q' \triangleleft_r Q} m'_{Q'Q} C'_{Q'}$ $m'_{Q'Q} \in uK_0 \cap u^{-1}K_\infty$).*

By [3, §4], the $\mathscr{H}_J$-module with basis $(\mathrm{Res}_{\mathscr{H}_J} M_\lambda, \Gamma'_\lambda)$ decomposes into right cells as

$$\Gamma'_\lambda = \bigsqcup_{i \in [k_\lambda]} \{C'_Q : \mathrm{sh}(Q|_{[r-1]}) = \lambda - a_i\},$$

and moreover, $\{C'_Q : \mathrm{sh}(Q|_{[r-1]}) = \lambda - a_i\} \xrightarrow{\cong} \Gamma'_{\lambda - a_i},\ C'_Q \mapsto C'_{Q|_{[r-1]}}$ is an isomorphism of right $\mathscr{H}_J$-cells.

Corollary 3.3. *Let $\leq_{\mathrm{Res}_J \Gamma'_\lambda}$ be the partial order on the right cells of the $\mathscr{H}_J$-module with basis $(\mathrm{Res}_{\mathscr{H}_J} M_\lambda, \Gamma'_\lambda)$. This partial order is a total order with $\Gamma'_{\lambda - a_i} \leq_{\mathrm{Res}_J \Gamma'_\lambda} \Gamma'_{\lambda - a_j}$ exactly when $i \leq j$. Similarly, $(\mathrm{Res}_{\mathscr{H}_J} M_\lambda, \Gamma_\lambda)$ has a right cell isomorphic to $\Gamma_{\lambda - a_i}$ for each $i \in [k_\lambda]$*

[2]Lemma 7.4 of [5] uses a different partial order, but the proof given for this lemma also works for the partial order $\triangleleft_r$ defined here.

and the partial order $\leq_{\mathrm{Res}_J \Gamma_\lambda}$ *on right cells is a total order with* $\Gamma_{\lambda - a_i} \leq_{\mathrm{Res}_J \Gamma_\lambda} \Gamma_{\lambda - a_j}$ *exactly when* $i \geq j$*.*

Proof. Lemma 3.2 shows that $\Gamma'_{\lambda - a_i} \leq_{\mathrm{Res}_J \Gamma'_\lambda} \Gamma'_{\lambda - a_j}$ implies $i \leq j$. To prove the converse, it suffices to show the existence of certain nonzero $\mu(Q', Q)$. The DKT_i-edges from (11) suffice. □

We will also need the following theorem, one of the main results of [5].

Theorem 3.4 ([5]). *The transition matrix expressing the lower canonical basis* $\{C'_Q : Q \in SYT(\lambda)\}$ *of* M_λ *in terms of the upper canonical basis* $\{C_Q : Q \in SYT(\lambda)\}$ *of* M_λ *has entries belonging to* $K_0 \cap K_\infty$ *and is the identity matrix at* $u = 0$ *and* $u = \infty$*.*

See [5, Example 7.5] for an example of this transition matrix. One consequence of this theorem is that $K_0\Gamma'_\lambda = K_0\Gamma_\lambda$. Let $\mathscr{L}_\lambda$ denote this K_0-lattice.

Lemma 3.5 (The projection lemma). *Fix* $i \in [k_\lambda]$*. Let* $x = \sum_{Q \in SYT(\lambda)} a_Q C'_Q$ *be an element of* M_λ *such that for each* Q *with* $\mathrm{sh}(Q|_{[r-1]}) = \lambda - a_j$ *and* $j \geq i$*, there holds* $a_Q \in K_0$*. Then*

$$p_{M_{\lambda - a_i}}(x) \equiv \sum_{\substack{Q \in SYT(\lambda), \\ Q_{a_i} = r}} a_Q (\tilde{C}'_Q)^J \mod u\mathscr{L}_{\lambda - a_i}.$$

Similarly, if $x = \sum_{Q \in SYT(\lambda)} a_Q C_Q$ *is an element of* M_λ *such that for each* Q *with* $\mathrm{sh}(Q|_{[r-1]}) = \lambda - a_j$ *and* $j \leq i$*, there holds* $a_Q \in K_0$*, then*

$$p_{M_{\lambda - a_i}}(x) \equiv \sum_{\substack{Q \in SYT(\lambda), \\ Q_{a_i} = r}} a_Q (\tilde{C}_Q)^J \mod u\mathscr{L}_{\lambda - a_i}.$$

Proof. This follows easily from Lemma 3.2. □

4. The nonstandard Hecke algebra $\breve{\mathscr{H}}_r$

The nonstandard Hecke algebra was introduced in [11] to study the Kronecker problem. Its role in the Kronecker problem is discussed in [2, §1] and [9]; some of its representation theory is discussed in [2, §11] and [10], including a complete description $K\breve{\mathscr{H}}_3$ and $K\breve{\mathscr{H}}_4$-irreducibles; the problem of constructing a canonical basis for $\breve{\mathscr{H}}_r$ is discussed and [2, §19] and [9]. The main purpose of this paper is to determine the irreducibles of the nonstandard Temperley-Lieb algebra $K\breve{\mathscr{H}}_{r,2}$, which is a quotient algebra of $K\breve{\mathscr{H}}_r$. Here we assemble some basic facts about $\breve{\mathscr{H}}_r$ from [4, 11, 2] and prove a few new ones.

4.1. **Definition of $\breve{\mathscr{H}}_r$.** Recall that S is now defined to be $\{s_1, \ldots, s_{r-1}\}$. We repeat the definition of $\breve{\mathscr{H}}_r$ from the introduction:

Definition 4.1. The *type A nonstandard Hecke algebra* $\breve{\mathscr{H}}_r$ is the subalgebra of $\mathscr{H}_r \otimes \mathscr{H}_r$ generated by the elements

$$\mathcal{P}_s := C'_s \otimes C'_s + C_s \otimes C_s, \ \ s \in S. \tag{12}$$

We let $\check{\Delta} : \check{\mathscr{H}}_r \hookrightarrow \mathscr{H}_r \otimes \mathscr{H}_r$ denote the canonical inclusion, which we think of as a deformation of the coproduct $\Delta_{\mathbb{Z}\mathcal{S}_r} : \mathbb{Z}\mathcal{S}_r \to \mathbb{Z}\mathcal{S}_r \otimes \mathbb{Z}\mathcal{S}_r$, $w \mapsto w \otimes w$.

The nonstandard Hecke algebra is also the subalgebra of $\mathscr{H}_r \otimes \mathscr{H}_r$ generated by

$$\mathcal{Q}_s := [2]^2 - \mathcal{P}_s = -C'_s \otimes C_s - C_s \otimes C'_s,\ s \in S.$$

We will write $\mathcal{P}_i$ (resp. $\mathcal{Q}_i$) as shorthand for $\mathcal{P}_{s_i}$ (resp. $\mathcal{Q}_{s_i}$), $i \in [r-1]$. For a ring homomorphism $K \to \mathbf{A}$, we have the specialization $K\check{\mathscr{H}}_r := K \otimes_{\mathbf{A}} \check{\mathscr{H}}_r$ of the nonstandard Hecke algebra.

The elements $\mathcal{P}_i$ and $\mathcal{Q}_i$ satisfy the quadratic relations $\mathcal{P}_i^2 = [2]^2\mathcal{P}_i$ and $\mathcal{Q}_i^2 = [2]^2\mathcal{Q}_i$, and $\mathcal{P}_i$ and $\mathcal{P}_{i+1}$ satisfy a nonstandard version of the braid relation (see [4]). For $r \geq 4$, the $\mathcal{P}_i$ satisfy additional relations which seem to be extremely difficult to describe (see [11]).

4.2. **Representation theory of $S^2\mathscr{H}_r$.** The representations of $\check{\mathscr{H}}_r$ are related to those of $S^2\mathscr{H}_r$ by the fact that $\check{\mathscr{H}}_r \subseteq S^2\mathscr{H}_r$ (see, e.g., [2, Proposition 11.6]), so any $S^2\mathscr{H}_r$-module is an $\check{\mathscr{H}}_r$-module by restriction. We recall the description of the $KS^2\mathscr{H}_r$-irreducibles from [2]. These irreducibles are close to those of $K\check{\mathscr{H}}_r$, and even closer to those of $K\check{\mathscr{H}}_{r,2}$, which will be described in §5.

First note that we have the following commutativity property for any $\mathscr{H}_r$-modules M and M':

$$\mathrm{Res}_{S^2\mathscr{H}_r} M \otimes M' \cong \mathrm{Res}_{S^2\mathscr{H}_r} M' \otimes M, \tag{13}$$

where the isomorphism is given by the flip τ, $\tau(a \otimes b) = b \otimes a$.

Recall from §2 that $\mathscr{P}_r$ denotes the set of partitions of size r and $\mathscr{P}'_r$ the set of partitions of r that are not a single row or column shape.

Proposition-Definition 4.2 ([2]). *Define the following $S^2\mathscr{H}_r$-modules. After tensoring these with K, this is the list of distinct $KS^2\mathscr{H}_r$-irreducibles*

(1) $M^{\mathbf{A}}_{\{\lambda,\mu\}} := \mathrm{Res}_{S^2\mathscr{H}_r} M^{\mathbf{A}}_\lambda \otimes M^{\mathbf{A}}_\mu$, $\{\lambda, \mu\} \subseteq \mathscr{P}_r$, $\lambda \neq \mu$,
(2) $S^2M^{\mathbf{A}}_\lambda := \mathrm{Res}_{S^2\mathscr{H}_r} S^2M^{\mathbf{A}}_\lambda$, $\lambda \in \mathscr{P}_r$,
(3) $\Lambda^2M^{\mathbf{A}}_\lambda := \mathrm{Res}_{S^2\mathscr{H}_r} \Lambda^2M^{\mathbf{A}}_\lambda$, $\lambda \in \mathscr{P}'_r$.

Let $M_{\{\lambda,\mu\}}$, S^2M_λ, Λ^2M_λ denote the corresponding $KS^2\mathscr{H}_r$-modules.

4.3. **Contragradients of $\mathscr{H}_r$-modules.** Any anti-automorphism S of an $\mathbf{A}$-algebra H allows us to define contragradients of H-modules: let $\langle\cdot,\cdot\rangle : M \otimes M^* \to \mathbf{A}$ be the canonical pairing, where M^* is the $\mathbf{A}$-module $\mathrm{Hom}_{\mathbf{A}}(M, \mathbf{A})$. Then the H-module structure on M^* is defined by

$$\langle m, m'h\rangle = \langle mS(h), m'\rangle \text{ for any } h \in H,\ m \in M, m' \in M^*.$$

There is an $\mathbf{A}$-algebra automorphism $\theta : \mathscr{H}_r \to \mathscr{H}_r$ defined by $\theta(T_s) = -T_s^{-1}$, $s \in S$. It is not hard to show that θ is an involution and satisfies $\theta(C'_w) = (-1)^{\ell(w)}C_w$. Let 1^{op} be the $\mathbf{A}$-anti-automorphism of $\mathscr{H}_r$ given by $1^{\mathrm{op}}(T_w) = T_{w^{-1}}$. Let θ^{op} be the $\mathbf{A}$-anti-automorphism of $\mathscr{H}_r$ given by $\theta^{\mathrm{op}} = \theta \circ 1^{\mathrm{op}} = 1^{\mathrm{op}} \circ \theta$.

Let $\{C'^{\vee}_w : w \in \mathcal{S}_r\} \subseteq \mathrm{Hom}_{\mathbf{A}}(\mathscr{H}_r, \mathbf{A})$ be the basis dual to $\{C'_w : w \in \mathcal{S}_r\}$. Let w_0 be the longest element of $\mathcal{S}_r$.

Let Z^*_λ be the SYT of shape λ with $1, \ldots, \lambda_1$ in the first row, $\lambda_1 + 1, \ldots, \lambda_1 + \lambda_2$ in the second row, etc. For an SYT Q, let $\ell(Q)$ denote the distance between Q and Z^*_λ in the DE graph on SYT(λ). It is not hard to show that for any $P \in$ SYT(λ), $\ell(Q) \equiv \ell(w) - \ell(z)$ mod 2, where $w = \mathrm{RSK}^{-1}(P, Q)$, $z = \mathrm{RSK}^{-1}(P, Z^*_\lambda)$.

Proposition 4.3. (i) *The right $\mathscr{H}_r$-modules $\mathscr{H}_r^\diamond$ and $\mathscr{H}_r$ are isomorphic via*

$$\alpha_\diamond : \mathscr{H}_r^\diamond \xrightarrow{\cong} \mathscr{H}_r,\ C'^{\vee}_w \mapsto C_{w_0 w},\ w \in \mathcal{S}_r.$$

(ii) *The right $\mathscr{H}_r$-modules $\mathscr{H}_r^\#$ and $\mathscr{H}_r$ are isomorphic via*

$$\alpha_\# : \mathscr{H}_r^\# \xrightarrow{\cong} \mathscr{H}_r,\ C'^{\vee}_w \mapsto (-1)^{\ell(w)} C'_{w_0 w},\ w \in \mathcal{S}_r.$$

(iii) *The restriction of $\alpha_\diamond^{-1}$ to any right cell Γ_λ of $\Gamma_{\mathcal{S}_r}$ yields the isomorphism*

$$M^{\mathbf{A}}_\lambda \xrightarrow{\cong} (M^{\mathbf{A}}_\lambda)^\diamond,\ C_Q \mapsto C'^{\vee}_Q,\ Q \in SYT(\lambda).$$

(iv) *The restriction of $\alpha_\#^{-1}$ to any right cell Γ'_λ of $\Gamma'_{\mathcal{S}_r}$ yields, up to a sign, the isomorphism*

$$M^{\mathbf{A}}_{\lambda'} \xrightarrow{\cong} (M^{\mathbf{A}}_\lambda)^\#,\ (-1)^{\ell(Q^T)} C'_Q \mapsto C'^{\vee}_{Q^T},\ Q \in SYT(\lambda').$$

Proof. We first record the following formulae which are immediate from (6), (7), and $R(w_0 w) = S \backslash R(w)$.

$$C_{w_0 w} C'_s = \begin{cases} [2] C_{w_0 w} + \sum_{\{w_0 w' \in \mathcal{S}_r : s \notin R(w')\}} \mu(w_0 w', w_0 w) C_{w_0 w'} & \text{if } s \in R(w), \\ 0 & \text{if } s \notin R(w). \end{cases} \tag{14}$$

$$C'_w \theta(C'_s) = \begin{cases} 0 & \text{if } s \in R(w), \\ [2] C'_w - \sum_{\{w' \in \mathcal{S}_r : s \in R(w')\}} \mu(w', w) C'_{w'} & \text{if } s \notin R(w). \end{cases} \tag{15}$$

By the definition of $\mathscr{H}_r^\diamond$,

$$C'^{\vee}_w C'_s = \begin{cases} [2] C'^{\vee}_w + \sum_{\{w' \in \mathcal{S}_r : s \notin R(w')\}} \mu(w, w') C'^{\vee}_{w'} & \text{if } s \in R(w), \\ 0 & \text{if } s \notin R(w). \end{cases} \tag{16}$$

Statement (i) then follows from (14) as $\mu(w, w') = \mu(w_0 w', w_0 w)$ [7, Corollary 3.2].

By the definition of $\mathscr{H}_r^\#$ and from (15), we obtain

$$C'^{\vee}_w C'_s = \begin{cases} [2] C'^{\vee}_w & \text{if } s \notin R(w), \\ -\sum_{\{w' \in \mathcal{S}_r : s \notin R(w')\}} \mu(w, w') C'^{\vee}_{w'} & \text{if } s \in R(w). \end{cases} \tag{17}$$

Statement (ii) then follows from (6) using $R(w_0 w) = S \backslash R(w)$, $\mu(w, w') = \mu(w_0 w', w_0 w)$, and the fact that $\mu(w', w) \neq 0$ implies $(-1)^{\ell(w')} = -(-1)^{\ell(w)}$.

Statements (iii) and (iv) then follow from (i) and (ii), respectively, the fact that $Q(w_0 w) = Q(w)^T$ (see, e.g., [6, A1.2]), and the definitions in §3.4. $\square$

As discussed in [4, 2], the inclusion $\check{\Delta} : \check{\mathscr{H}}_r \hookrightarrow \mathscr{H}_r \otimes \mathscr{H}_r$ is not a good approximation of the coproduct $\Delta_{\mathbb{Z}\mathcal{S}_r}$, though it is in a certain sense the closest approximation possible. There are a couple ways that $\check{\mathscr{H}}_r$ behaves like a Hopf algebra, one of which is the following.

Proposition 4.4 ([4]). *The involutions 1^{op} and θ^{op} are antipodes in the following sense:*

$$\mu \circ (1^{op} \otimes 1) \circ \check{\Delta} = \eta \circ \check{\epsilon}_+,$$
$$\mu \circ (\theta^{op} \otimes 1) \circ \check{\Delta} = \eta \circ \check{\epsilon}_-,$$

where these are equalities of maps from $\check{\mathscr{H}}_r$ to $\mathscr{H}_r$. Here μ is the multiplication map for $\mathscr{H}_r$ and $\eta : K \to \mathscr{H}_r$ is the unit of $\mathscr{H}_r$.

4.4. **Some representation theory of $\check{\mathscr{H}}_r$.** It is shown in [2] (Proposition 11.8) that $K\check{\mathscr{H}}_r$ is semisimple.

Remark 4.5. It is reasonable to suspect that $K\check{\mathscr{H}}_r$ is split semisimple, and indeed, our computations are consistent with this being true. In this paper we show that the nonstandard Temperley-Lieb algebra $K\check{\mathscr{H}}_{r,2}$ is split semisimple by explicitly determining its irreducibles. We are curious if there is a way to show that $K\check{\mathscr{H}}_r$ is split semisimple without explicitly determining its irreducibles.

There are one-dimensional trivial and sign representations of $\check{\mathscr{H}}_r$, which we denote by $\check{\epsilon}_+$ and $\check{\epsilon}_-$:

$$\check{\epsilon}_+ : \mathcal{P}_s \mapsto [2]^2, \qquad \check{\epsilon}_- : \mathcal{P}_s \mapsto 0, \quad s \in S.$$

For $\lambda, \mu \vdash r$, the $\check{\mathscr{H}}_r$-module $\operatorname{Res}_{\check{\mathscr{H}}_r} M_\lambda^{\mathbf{A}} \otimes M_\mu^{\mathbf{A}} \cong \operatorname{Res}_{\check{\mathscr{H}}_r} M_\mu^{\mathbf{A}} \otimes M_\lambda^{\mathbf{A}}$ is denoted[3] $\check{M}_{\lambda,\mu}^{\mathbf{A}}$. Let $\check{S}^2\check{M}_\lambda^{\mathbf{A}}$ (resp. $\check{\Lambda}^2\check{M}_\lambda^{\mathbf{A}}$) denote the $\check{\mathscr{H}}_r$-module $\operatorname{Res}_{\check{\mathscr{H}}_r} S^2 M_\lambda^{\mathbf{A}}$ (resp. $\operatorname{Res}_{\check{\mathscr{H}}_r} \Lambda^2 M_\lambda^{\mathbf{A}}$), where $S^2 M_\lambda^{\mathbf{A}}$ and $\Lambda^2 M_\lambda^{\mathbf{A}}$ are as in Proposition-Definition 4.2. Let $\check{M}_{\lambda,\mu}, \check{S}^2\check{M}_\lambda, \check{\Lambda}^2\check{M}_\lambda$ denote the corresponding $K\check{\mathscr{H}}_r$-modules.

Let

$$\mathbf{A} \xrightarrow{I} (M_\lambda^{\mathbf{A}})^\diamond \otimes M_\lambda^{\mathbf{A}} \tag{18}$$

be the canonical inclusion given by sending $1 \in \mathbf{A}$ to $I \in \operatorname{End}(M_\lambda^{\mathbf{A}}) \cong (M_\lambda^{\mathbf{A}})^\diamond \otimes M_\lambda^{\mathbf{A}}$. Let

$$M_\lambda^{\mathbf{A}} \otimes (M_\lambda^{\mathbf{A}})^\diamond \xrightarrow{\mathrm{tr}} \mathbf{A} \tag{19}$$

be the canonical surjection.

We then have the following $\check{\mathscr{H}}_r$-module homomorphisms

$$\check{\epsilon}_+ \xrightarrow{I} (M_\lambda^{\mathbf{A}})^\diamond \otimes M_\lambda^{\mathbf{A}},$$
$$\ker(\mathrm{tr}) \hookrightarrow M_\lambda^{\mathbf{A}} \otimes (M_\lambda^{\mathbf{A}})^\diamond \xrightarrow{\mathrm{tr}} \check{\epsilon}_+,$$

To see this, note that in general, if M is an H-module and H is a Hopf algebra with counit ϵ, then it follows from the axiom for the antipode that $\epsilon \xrightarrow{I} M^* \otimes M$ and $M \otimes M^* \xrightarrow{\mathrm{tr}} \epsilon$ are H-module homomorphisms. The same proof works in the present setting using Proposition 4.4 in place of the antipode axiom.

Since $\frac{1}{|\mathrm{SYT}(\lambda)|}\tau \circ I$ is a splitting of tr and $(M_\lambda^{\mathbf{A}})^\diamond \cong M_\lambda^{\mathbf{A}}$ (Proposition 4.3 (iii)), we obtain the decomposition of $\check{\mathscr{H}}_r$-modules

$$\ker(\mathrm{tr}) \oplus \check{\epsilon}_+ \cong \check{M}_{\lambda,\lambda}^{\mathbf{A}}. \tag{20}$$

[3]The more correct notation $\check{M}_{\{\lambda,\mu\}}^{\mathbf{A}}$ is used in [2], but in this paper the shorter $\check{M}_{\lambda,\mu}^{\mathbf{A}}$ is preferable for being less cumbersome.

Moreover, as a consequence of Proposition 4.6 (i) below, $\check{\epsilon}_+ \subseteq \check{M}^{\mathbf{A}}_{\lambda,\lambda}$ lies in $\check{S}^2\check{M}^{\mathbf{A}}_\lambda$. Then define $S'\check{M}^{\mathbf{A}}_\lambda := \ker(\mathrm{tr}) \cap \check{S}^2\check{M}^{\mathbf{A}}_\lambda$. The decomposition (20) yields the decomposition

$$\check{S}^2\check{M}^{\mathbf{A}}_\lambda \cong S'\check{M}^{\mathbf{A}}_\lambda \oplus \check{\epsilon}_+. \tag{21}$$

Proposition 4.6. *The maps* (18) *and* (19) *as well as the analogous maps for $\check{\epsilon}_-$ can be made explicit using upper and lower canonical bases:*

(i) *The inclusion $\check{\epsilon}_+ \hookrightarrow M^{\mathbf{A}}_\lambda \otimes M^{\mathbf{A}}_\lambda$ is given by*

$$1 \mapsto \sum_{Q\in SYT(\lambda)} C_Q \otimes C'_Q = \sum_{Q\in SYT(\lambda)} C'_Q \otimes C_Q.$$

(ii) *The surjection $M^{\mathbf{A}}_\lambda \otimes M^{\mathbf{A}}_\lambda \to \check{\epsilon}_+$ is given by*

$$\sum_{T,U\in SYT(\lambda)} a^{TU} C'_T \otimes C_U \mapsto \tfrac{1}{|SYT(\lambda)|} \sum_{U\in SYT(\lambda)} a^{UU}, \text{ for any } a^{TU} \in \mathbf{A}.$$

(iii) *The inclusion $\check{\epsilon}_- \hookrightarrow M^{\mathbf{A}}_{\lambda'} \otimes M^{\mathbf{A}}_\lambda$ is given by*

$$1 \mapsto \sum_{Q\in SYT(\lambda)} (-1)^{\ell(Q)} C'_{Q^T} \otimes C'_Q.$$

(iv) *The surjection $M^{\mathbf{A}}_\lambda \otimes M^{\mathbf{A}}_{\lambda'} \to \check{\epsilon}_-$ is given by*

$$\sum_{T,U\in SYT(\lambda)} a^{TU^T} C'_T \otimes C'_{U^T} \mapsto \tfrac{1}{|SYT(\lambda)|} \sum_{U\in SYT(\lambda)} (-1)^{\ell(U)} a^{UU^T}, \text{ for any } a^{TU^T} \in \mathbf{A}.$$

Note that since $\check{\epsilon}_+ \subseteq \check{S}^2\check{M}^{\mathbf{A}}_\lambda$ by (i), (ii) remains valid with $C_U \otimes C'_T$ in place of $C'_T \otimes C_U$.

Proof. The map I of (18) is given by

$$1 \mapsto \textstyle\sum_{Q\in \mathrm{SYT}(\lambda)} {C'_Q}^\vee \otimes C'_Q.$$

Applying the isomorphism of Proposition 4.3 (iii) then yields (i), except the equality. The equality in (i) follows from the fact that $\tau \circ I : \check{\epsilon}_+ \hookrightarrow M^{\mathbf{A}}_\lambda \otimes (M^{\mathbf{A}}_\lambda)^\diamond$ is an $\check{\mathscr{H}}_r$-module homomorphism (since $\check{\mathscr{H}}_r \subseteq S^2\mathscr{H}_r$), the multiplicity of $K\check{\epsilon}_+$ in $\check{M}_{\lambda,\lambda}$ is 1, and Theorem 3.4.

The map tr of (19) is given by

$$\textstyle\sum_{T,U\in \mathrm{SYT}(\lambda)} a^{TU} C'_T \otimes {C'_U}^\vee \mapsto \tfrac{1}{|\mathrm{SYT}(\lambda)|} \sum_U a^{UU}, \text{ for any } a^{TU} \in \mathbf{A},$$

so (ii) also follows from Proposition 4.3 (iii). Statements (iii) and (iv) are proved in a similar way using Proposition 4.4 and Proposition 4.3 (iv). $\square$

4.5. **The action of $\mathcal{P}_s$ on $M_\lambda \otimes M_\mu$.** For the proof of the main theorem, it is convenient to record the action of $\mathcal{P}_s$ on $M_\lambda \otimes M_\mu$ in the bases

$$\Gamma'_\lambda \otimes \Gamma'_\mu = \{C'_T \otimes C'_U : T \in \mathrm{SYT}(\lambda),\ U \in \mathrm{SYT}(\mu)\},$$

$\Gamma_\lambda \otimes \Gamma'_\mu$, and $\Gamma_\lambda \otimes \Gamma_\mu$. These calculations are easily made using (6) and (7).

$$(C'_T \otimes C'_U)\mathcal{P}_s = \tag{22}$$
$$\begin{cases} [2]^2 C'_T \otimes C'_U & \text{if } s \in R(C'_T) \text{ and } s \in R(C'_U) \\ [2] \displaystyle\sum_{s \in R(C'_{U'})} \mu(U',U) C'_T \otimes C'_{U'} & \text{if } s \in R(C'_T) \text{ and } s \notin R(C'_U) \\ [2] \displaystyle\sum_{s \in R(C'_{T'})} \mu(T',T) C'_{T'} \otimes C'_U & \text{if } s \notin R(C'_T) \text{ and } s \in R(C'_U) \\ [2]^2 C'_T \otimes C'_U & \\ -[2] \left(\displaystyle\sum_{s \in R(C'_{T'})} \mu(T',T) C'_{T'} \otimes C'_U + \displaystyle\sum_{s \in R(C'_{U'})} \mu(U',U) C'_T \otimes C'_{U'} \right) & \\ +2 \displaystyle\sum_{s \in R(C'_{T'}), s \in R(C'_{U'})} \mu(T',T)\mu(U',U) C'_{T'} \otimes C'_{U'} & \text{if } s \notin R(C'_U) \text{ and } s \notin R(C'_T) \end{cases}$$

$$(C_T \otimes C'_U)\mathcal{P}_s = \begin{cases} 0 & \text{if } s \in R(C_T) \text{ and } s \in R(C'_U) \\ [2]^2 C_T \otimes C'_U - [2] \sum_{s \in R(C'_{U'})} \mu(U',U) C_T \otimes C'_{U'} & \text{if } s \in R(C_T) \text{ and } s \notin R(C'_U) \\ [2]^2 C_T \otimes C'_U + [2] \sum_{s \in R(C_{T'})} \mu(T',T) C_{T'} \otimes C'_U & \text{if } s \notin R(C_T) \text{ and } s \in R(C'_U) \\ -[2] \sum_{s \in R(C_{T'})} \mu(T',T) C_{T'} \otimes C'_U & \\ +[2] \sum_{s \in R(C'_{U'})} \mu(U',U) C_T \otimes C'_{U'} & \\ +2 \sum_{s \in R(C_{T'}), s \in R(C'_{U'})} \mu(T',T)\mu(U',U) C_{T'} \otimes C'_{U'} & \text{if } s \notin R(C_T) \text{ and } s \notin R(C'_U) \end{cases} \tag{23}$$

$$(C_T \otimes C_U)\mathcal{P}_s = \tag{24}$$
$$\begin{cases} [2]^2 C_T \otimes C_U & \text{if } s \in R(C_T) \text{ and } s \in R(C_U) \\ -[2] \displaystyle\sum_{s \in R(C_{U'})} \mu(U',U) C_T \otimes C_{U'} & \text{if } s \in R(C_T) \text{ and } s \notin R(C_U) \\ -[2] \displaystyle\sum_{s \in R(C_{T'})} \mu(T',T) C_{T'} \otimes C_U & \text{if } s \notin R(C_T) \text{ and } s \in R(C_U) \\ [2]^2 C_T \otimes C_U & \\ +[2] \left(\displaystyle\sum_{s \in R(C_{T'})} \mu(T',T) C_{T'} \otimes C_U + \displaystyle\sum_{s \in R(C_{U'})} \mu(U',U) C_T \otimes C_{U'} \right) & \\ +2 \displaystyle\sum_{s \in R(C_{T'}), s \in R(C_{U'})} \mu(T',T)\mu(U',U) C_{T'} \otimes C_{U'} & \text{if } s \notin R(C_U) \text{ and } s \notin R(C_T) \end{cases}$$

5. Irreducibles of $\check{\mathscr{H}}_{r,2}$

Define the *Temperley-Lieb* algebra $\mathscr{H}_{r,d}$ to be the quotient of $\mathscr{H}_r$ by the two-sided ideal

$$\bigoplus_{\substack{\lambda \vdash r,\ \ell(\lambda)>d,\\ P\in \mathrm{SYT}(\lambda)}} \mathbf{A}\Gamma_P = \mathbf{A}\{C_w : \ell(\mathrm{sh}(P(w))) > d\}.$$

Define the *nonstandard Temperley-Lieb* algebra $\check{\mathscr{H}}_{r,d}$ to be the subalgebra of $\mathscr{H}_{r,d} \otimes \mathscr{H}_{r,d}$ generated by the elements $\mathcal{P}_s := C'_s \otimes C'_s + C_s \otimes C_s,\ s \in S$.

Let $\mathscr{P}_{r,2}$ be the set of partitions of size r with at most two parts and $\mathscr{P}'_{r,2}$ be the subset of $\mathscr{P}_{r,2}$ consisting of those partitions that are not a single row or column shape. Define the index set $\check{\mathscr{P}}_{r,2}$ for the $K\check{\mathscr{H}}_{r,2}$-irreducibles as follows:

$$\check{\mathscr{P}}_{r,2} = \{\{\lambda,\mu\} : \lambda,\mu \in \mathscr{P}_{r,2},\ \lambda \neq \mu\} \sqcup \{+\lambda : \lambda \in \mathscr{P}'_{r,2}\} \sqcup \{-\lambda : \lambda \in \mathscr{P}'_{r,2}\} \sqcup \{\check{\epsilon}_+\}. \quad (25)$$

This section is devoted to a proof of the main result of this paper:

Theorem 5.1. *The algebra $K\check{\mathscr{H}}_{r,2}$ is split semisimple and the list of distinct irreducibles is*

(1) $\check{M}_\alpha := \check{M}_{\lambda,\mu} = \mathrm{Res}_{K\check{\mathscr{H}}_{r,2}} M_\lambda \otimes M_\mu$, *for* $\alpha = \{\lambda,\mu\} \in \check{\mathscr{P}}_{r,2}$,
(2) $\check{M}_\alpha := S'\check{M}_\lambda$, *for* $\alpha = +\lambda \in \check{\mathscr{P}}_{r,2}$,
(3) $\check{M}_\alpha := \check{\Lambda}^2\check{M}_\lambda$, *for* $\alpha = -\lambda \in \check{\mathscr{P}}_{r,2}$,
(4) $\check{M}_\alpha := K\check{\epsilon}_+$, *for* $\alpha = \check{\epsilon}_+ \in \check{\mathscr{P}}_{r,2}$.

Moreover, the irreducible $K(\mathscr{H}_{r,2} \otimes \mathscr{H}_{r,2})$-modules decompose into $K\check{\mathscr{H}}_{r,2}$-irreducibles as follows

$$\begin{array}{ll} M_\lambda \otimes M_\mu \cong \check{M}_{\lambda,\mu} & \text{if } \lambda \neq \mu, \\ M_\lambda \otimes M_\lambda \cong S'\check{M}_\lambda \oplus \check{\Lambda}^2\check{M}_\lambda \oplus K\check{\epsilon}_+ & \lambda \vdash r. \end{array}$$

5.1. Gluing $K\check{\mathscr{H}}_{r-1}$-irreducibles.

Proposition 5.2. *The four types of $K\check{\mathscr{H}}_r$-modules from Theorem 5.1 decompose into $K\check{\mathscr{H}}_{r-1}$-modules as follows:*

(1a) $\mathrm{Res}_{K\check{\mathscr{H}}_{r-1}} \check{M}_{\lambda,\mu} \cong \bigoplus_{i\in[k_\lambda], j\in[k_\mu]} \check{M}_{\lambda-a_i,\mu-b_j}$, *if* $|\lambda \cap \mu| < r-1$.
(1b) $\mathrm{Res}_{K\check{\mathscr{H}}_{r-1}} \check{M}_{\lambda,\mu} \cong \bigoplus_{\substack{i\in[k_\lambda], j\in[k_\mu],\\ (i,j)\neq(k,l)}} \check{M}_{\lambda-a_i,\mu-b_j} \oplus S'\check{M}_\nu \oplus \check{\Lambda}^2\check{M}_\nu \oplus K\check{\epsilon}_+$, *where* $\nu = \lambda - a_k = \mu - b_l$.
(2) $\mathrm{Res}_{K\check{\mathscr{H}}_{r-1}} S'\check{M}_\lambda \cong \bigoplus_{1\leq i<j\leq k_\lambda} \check{M}_{\lambda-a_i,\lambda-a_j} \oplus \bigoplus_{i\in[k_\lambda]} S'\check{M}_{\lambda-a_i} \oplus K\check{\epsilon}_+^{\oplus k_\lambda - 1}$, *for* $\lambda \in \mathscr{P}'_r$.
(3) $\mathrm{Res}_{K\check{\mathscr{H}}_{r-1}} \check{\Lambda}^2\check{M}_\lambda \cong \bigoplus_{1\leq i<j\leq k_\lambda} \check{M}_{\lambda-a_i,\lambda-a_j} \oplus \bigoplus_{i\in[k_\lambda]} \check{\Lambda}^2\check{M}_{\lambda-a_i}$, *for* $\lambda \in \mathscr{P}'_r$.
(4) $\mathrm{Res}_{K\check{\mathscr{H}}_{r-1}} K\check{\epsilon}_+ \cong K\check{\epsilon}_+$.

Note that if $|\mathrm{SYT}(\nu)| = 1$, then $S'\check{M}_\nu = \check{\Lambda}^2\check{M}_\nu = 0$, so some zero modules appear in the right-hand sides.

Proof. It is well known that $\mathrm{Res}_{\mathscr{H}_{r-1}} M_\lambda \cong \bigoplus_{i\in[k_\lambda]} M_{\lambda-a_i}$. The decompositions (1a) and (1b) are clear from this and (21). Decomposition (3) follows from the general fact that $\Lambda(\bigoplus_{i\in[k]} M_i) \cong \Lambda(M_1)\otimes\ldots\otimes\Lambda(M_k)$ is a graded isomorphism of algebras for any vector spaces $M_1,\ldots,M_k$, where $\Lambda(M)$ is the exterior algebra of M. The analogous fact holds for symmetric algebras, which implies

$$\mathrm{Res}_{KS^2\mathscr{H}_{r-1}} S^2 M_\lambda \cong \bigoplus_{1\le i<j\le k_\lambda} \check{M}_{\lambda-a_i,\lambda-a_j} \oplus \bigoplus_{i\in[k_\lambda]} S^2 M_{\lambda-a_i}.$$

Decomposition (2) then follows from (21). $\square$

We adopt the convention that restrictions from $\check{\mathscr{H}}_r$ to $\check{\mathscr{H}}_{r-1}$ are considered with respect to the subalgebra of $\check{\mathscr{H}}_r$ generated by $\mathcal{P}_s$, $s\in J$, where $J := \{s_1,\ldots,s_{r-2}\}$. Given a $K\check{\mathscr{H}}_{r,d}$-irreducible N and a $K\check{\mathscr{H}}_{r,d}$-module M, let $\check{p}^0_N : M\to M$ denote the $K\check{\mathscr{H}}_{r,d}$ projector with image the N-isotypic component of M. Given a $K\check{\mathscr{H}}_{r-1,d}$-irreducible N and a $K\check{\mathscr{H}}_r$-module M, let $\check{p}^1_N : M\to M$ denote the $K\check{\mathscr{H}}_{r-1,d}$ projector with image the N-isotypic component of $\mathrm{Res}_{K\check{\mathscr{H}}_{r-1,d}} M$.

Theorem 5.1 will be proved inductively, using the list of $K\check{\mathscr{H}}_{r-1,2}$-irreducibles and the fact that the restriction of a $K\check{\mathscr{H}}_{r,2}$-irreducible to $K\check{\mathscr{H}}_{r-1,2}$ is multiplicity-free. Let $\bigoplus_{i\in[k]}\check{M}_i$ be a multiplicity-free decomposition of a $K\check{\mathscr{H}}_r$-module $\check{M}$ into $K\check{\mathscr{H}}_{r-1}$-irreducibles. Then any $K\check{\mathscr{H}}_r$-submodule of $\check{M}$ is a direct sum of some of the $\check{M}_i$. Suppose that $\check{M}' = \bigoplus_{i\in I}\check{M}_i$, for some $I\subseteq[k]$, is contained in a $K\check{\mathscr{H}}_r$-submodule of $\check{M}$. We say that $\check{M}'$ *glues* to $\check{M}_j$, $j\notin I$, if $\check{M}_j\subseteq \check{M}'(K\check{\mathscr{H}}_r)$; this is equivalent to $\check{p}^1_{\check{M}_j}(x)\neq 0$ for some $x\in\check{M}'(K\check{\mathscr{H}}_r)$. Thus if we show that $\check{M}_1$ glues to $\check{M}_2$, $\check{M}_1\oplus\check{M}_2$ glues to $\check{M}_3$, …, $\bigoplus_{i\in[k-1]}\check{M}_i$ glues to $\check{M}_k$, then this proves that $\check{M}$ is a $K\check{\mathscr{H}}_r$-irreducible. Slight variants of this argument will be used in the propositions in the next subsection.

5.2. **Four propositions on the irreducibility of $K\check{\mathscr{H}}_r$-modules.** In this subsection we state and prove Propositions 5.3, 5.5, 5.7, and 5.8, which will be used inductively to show that the $K\check{\mathscr{H}}_{r,2}$-modules in (1)–(4) of Theorem 5.1 are irreducible.

Proposition 5.3. *Maintain the setup of §5.1. If $\lambda\neq\mu$ and $\check{M}_{\lambda-a_i,\mu-b_j}$ are distinct irreducible $K\check{\mathscr{H}}_{r-1}$-modules ($i\in[k_\lambda], j\in[k_\mu]$), then $\check{M}_{\lambda,\mu}$ is an irreducible $K\check{\mathscr{H}}_r$-module.*

Proof. We work with the basis $\Gamma'_\lambda\otimes\Gamma'_\lambda$ of $\check{M}_{\lambda,\mu}$.

It suffices to show that $\check{M}_{\lambda-a_1,\mu-b_1}$ glues to $\check{M}_{\lambda-a_i,\mu-b_j}$ for $(i,j)\neq(1,1)$, which we do as follows: choose $T\in\mathrm{SYT}(\lambda)$ and $U\in\mathrm{SYT}(\mu)$ so that

$$\begin{aligned}&(1)\ T_{a_1}=r,\text{ and if } i\neq 1 \text{ then there is an edge } T\underset{r-1}{\leftrightsquigarrow}T' \text{ with } T'_{a_i}=r.\\&(2)\ U_{b_1}=r,\text{ and if } j\neq 1 \text{ then there is an edge } U\underset{r-1}{\leftrightsquigarrow}U' \text{ with } U'_{b_j}=r.\end{aligned}\tag{26}$$

Such tableaux exist by (11). Then if $i\neq1$ and $j\neq1$, then $s_{r-1}\notin R(C'_T)$, $s_{r-1}\notin R(C'_U)$ and $C'_T\otimes C'_U\mathcal{P}_{r-1}$ is computed using the last case of (22). The term $C'_{T'}\otimes C'_{U'}$ appears in the sum. The projection lemma (Lemma 3.5) and the fact that

$$\{(\tilde{C}'_A)^J\otimes(\tilde{C}'_B)^J : A\in\mathrm{SYT}(\lambda),\ A_{a_i}=r,\ B\in\mathrm{SYT}(\mu),\ B_{b_j}=r\}\cong\Gamma'_{\lambda-a_i}\otimes\Gamma'_{\mu-b_j}$$

is a K_0-basis of $\mathscr{L}_{\lambda-a_i} \otimes \mathscr{L}_{\mu-b_j}$ shows that the projection of $C'_T \otimes C'_U \mathcal{P}_{r-1}$ onto $\check{M}_{\lambda-a_i,\mu-b_j}$ is nonzero (the hypotheses of the lemma are satisfied, which depends in a somewhat delicate way on the form of the last case of (22)). Since $C'_T \otimes C'_U \in \check{M}_{\lambda-a_1,\mu-b_1}$ by Corollary 3.3, $\check{M}_{\lambda-a_1,\mu-b_1}$ glues to $\check{M}_{\lambda-a_i,\mu-b_j}$. If $i = 1$ or $j = 1$, these also glue by the same argument, possibly using the second or third case of (22) instead of the fourth. $\square$

Given a vector space M, let $\tau : M \otimes M \to M \otimes M$ denote the flip $a \otimes b \mapsto b \otimes a$. For $a, b \in M$, put $a \cdot b = \frac{1}{2}(1+\tau)(a \otimes b) = \frac{1}{2}(a \otimes b + b \otimes a)$ and $a \wedge b = \frac{1}{2}(1-\tau)(a \otimes b) = \frac{1}{2}(a \otimes b - b \otimes a)$.

Let $\mathscr{L}_\nu = K_0\Gamma'_\nu = K_0\Gamma_\nu$ be as defined after Theorem 3.4. Let $\leq$ be a total order on $\mathrm{SYT}(\lambda)$. Then

$$S^2\Gamma'_\nu := \Gamma'_\nu \cdot \Gamma'_\nu = \{C'_A \cdot C'_B : A, B \in \mathrm{SYT}(\nu),\ A \leq B\}$$

is a basis of S^2M_ν. Let $S^2\mathscr{L}_\nu := K_0 S^2\Gamma'_\nu$ be the corresponding K_0-lattice of S^2M_ν.

Similarly,

$$\Lambda^2\Gamma'_\nu := \{C'_A \wedge C'_B : A, B \in \mathrm{SYT}(\nu),\ A < B\}$$

is a basis of Λ^2M_ν. Let $\Lambda^2\mathscr{L}_\nu := K_0\Lambda^2\Gamma'_\nu$ be the corresponding K_0-lattice of Λ^2M_ν.

Lemma 5.4. *Fix some $T \in SYT(\nu)$. The set*

$$\{\check{p}^0_{S'\check{M}_\nu}(C'_A \cdot C'_B) : A, B \in SYT(\nu),\ A < B\} \sqcup \{\check{p}^0_{S'\check{M}_\nu}(C'_A \cdot C'_A) : A \in SYT(\nu),\ A \neq T\}$$

is a basis of $S'\check{M}_\nu$.

Proof. By Proposition 4.6 (i), $K\check{\epsilon}_+ \subseteq S^2M_\nu \subseteq M_\nu \otimes M_\nu$ is spanned by

$$\sum_{Q \in \mathrm{SYT}(\nu)} C_Q \otimes C'_Q \equiv \sum_{Q \in \mathrm{SYT}(\nu)} C'_Q \otimes C'_Q \mod uS^2\mathscr{L}_\nu, \tag{27}$$

where the equivalence is by Theorem 3.4.

As $S^2\Gamma'_\nu$ is a basis of S^2M_ν, to prove the lemma, it suffices to show that the left-hand side of (27) is not in the span of $S^2\Gamma'_\nu \setminus \{C'_T \cdot C'_T\}$. And this is true because the image of $\sum_{Q \in \mathrm{SYT}(\nu)} C'_Q \otimes C'_Q$ in $S^2\mathscr{L}_\nu/uS^2\mathscr{L}_\nu$ is not in the span of the image of $S^2\Gamma'_\nu \setminus \{C'_T \cdot C'_T\}$ in $S^2\mathscr{L}_\nu/uS^2\mathscr{L}_\nu$. $\square$

Proposition 5.5. *Maintain the setup of §5.1 and set $\nu = \lambda - a_k = \mu - b_l$. If the decomposition*

$$\mathrm{Res}_{K\check{\mathscr{H}}_{r-1}} \check{M}_{\lambda,\mu} \cong \bigoplus_{\substack{i \in [k_\lambda], j \in [k_\mu], \\ (i,j) \neq (k,l)}} \check{M}_{\lambda-a_i,\mu-b_j} \oplus S'\check{M}_\nu \oplus \check{\Lambda}^2\check{M}_\nu \oplus K\check{\epsilon}_+$$

of Proposition 5.2 (1b) consists of distinct irreducible $K\check{\mathscr{H}}_{r-1}$-modules, then $\check{M}_{\lambda,\mu}$ is an irreducible $K\check{\mathscr{H}}_r$-module.

Proof. First, if $k_\lambda = k_\mu = 1$, then $\lambda = (2)$ and $\mu = (1,1)$, and the result is clear in this case. We will then assume $(k,l) \neq (1,1)$ and prove the proposition using the basis $\Gamma'_\lambda \otimes \Gamma'_\lambda$;

if $(k,l)=(1,1)$, the proposition can be proved in a similar way[4] using the argument below with a_{k_λ}, b_{k_μ} in place of a_1, b_1 and the basis $\Gamma_\lambda \otimes \Gamma_\lambda$ in place of $\Gamma'_\lambda \otimes \Gamma'_\lambda$.

The $K\check{\mathscr{H}}_{r-1}$-irreducible $\check{M}_{\lambda-a_1,\mu-b_1}$ glues to $\check{M}_{\lambda-a_i,\mu-b_j}$ for $(i,j)\notin\{(k,l),(1,1)\}$ by the same argument as in the proof of Proposition 5.3.

The assumption $\lambda \rhd \mu$ implies $k \geq l$. Thus $k>1$ since we are assuming $(k,l)\neq(1,1)$. We will next show that $\bigoplus_{i\leq l}\check{M}_{\lambda-a_1,\mu-b_i}$ glues to $S'\check{M}_\nu$ and $\check{\Lambda}^2\check{M}_\nu$. We may assume that $|\mathrm{SYT}(\nu)|>1$ because this is equivalent to $S'\check{M}_\nu$ and $\check{\Lambda}^2\check{M}_\nu$ being nonzero. Thus by (11), we can choose $T,T'\in\mathrm{SYT}(\lambda)$ and $U\in\mathrm{SYT}(\mu)$ such that

(1) $T_{a_1}=r$ and there is an edge $T \underset{r-1}{\leftrightsquigarrow} T'$ with $T'_{a_k}=r$.
(2) $U_{b_l}=r$ and $U_\nu\neq T'_\nu$.

Here U_ν denotes the subtableau of U obtained by restricting U to ν. The quantity $C'_T\otimes C'_U\mathcal{P}_{r-1}$ is computed using the second or fourth case of (22): if the second case applies, then the projection lemma shows that

$$\check{p}^1_{M_\nu\otimes M_\nu}(C'_T\otimes C'_U\tfrac{\mathcal{P}_{r-1}}{[2]}) \equiv \sum_{\substack{s_{r-1}\in R(C'_A),\\ A_{a_k}=r}} \mu(A,T)(\tilde{C}'_A)^J\otimes(\tilde{C}'_U)^J \mod u\mathscr{L}_\nu\otimes\mathscr{L}_\nu; \tag{28}$$

if the fourth case applies, then a careful application of the projection lemma shows that

$$\check{p}^1_{M_\nu\otimes M_\nu}(C'_T\otimes C'_U\tfrac{\mathcal{P}_{r-1}}{[2]}) \equiv -\sum_{\substack{s_{r-1}\in R(C'_A),\\ A_{a_k}=r}} \mu(A,T)(\tilde{C}'_A)^J\otimes(\tilde{C}'_U)^J \mod u\mathscr{L}_\nu\otimes\mathscr{L}_\nu. \tag{29}$$

Let x (resp. $-x$) denote the right-hand side of (28) (resp. (29)). Since $\pm(\tilde{C}'_{T'})^J\otimes(\tilde{C}'_U)^J$ appears in the expression for $\pm x$ and $T'_\nu\neq U_\nu$, it follows that the projection of $\pm x$ to $\Lambda^2\mathscr{L}_\nu$ is nonzero. This uses that

$$\{(\tilde{C}'_A)^J\wedge(\tilde{C}'_B)^J : A\in\mathrm{SYT}(\lambda),\ A_{a_k}=r,\ B\in\mathrm{SYT}(\mu),\ B_{b_l}=r,\ A_\nu<B_\nu\}\cong\Lambda^2\Gamma'_\nu$$

is a K_0-basis of $\Lambda^2\mathscr{L}_\nu$. The quantities $\pm x$ also have nonzero projection onto $S'\check{M}_\nu$ by Lemma 5.4. Finally, we need that $C'_T\otimes C'_U\in\bigoplus_{i\leq l}\check{M}_{\lambda-a_1,\mu-b_i}$, which holds by Corollary 3.3, to conclude that $\bigoplus_{i\leq l}\check{M}_{\lambda-a_1,\mu-b_i}$ glues to $S'\check{M}_\nu$ and $\check{\Lambda}^2\check{M}_\nu$.

It remains to show that $K\check{\epsilon}_+\subseteq\check{M}_{\lambda-a_k,\mu-b_l}$ glues to some other $K\check{\mathscr{H}}_{r-1}$-irreducible of $\mathrm{Res}_{K\check{\mathscr{H}}_{r-1}}\check{M}_{\lambda,\mu}$. If not, then it follows that $\check{\epsilon}_+|_{u=1}$ is a 1-dimensional $\mathbb{Q}\mathcal{S}_r$-submodule of $\check{M}_{\lambda,\mu}|_{u=1}\cong M_\lambda|_{u=1}\otimes M_\mu|_{u=1}$. Here, the specialization $N|_{u=1}$ of an $\mathbf{A}$-module $N_\mathbf{A}$ is defined to be $\mathbb{Q}\otimes_\mathbf{A} N_\mathbf{A}$, the map $\mathbf{A}\to\mathbb{Q}$ given by $u\mapsto 1$. We are assuming $r\geq 3$, so $\check{\epsilon}_+\mathcal{P}_1=[2]^2\check{\epsilon}_+$. But then $\check{\epsilon}_+|_{u=1}$ is the trivial $\mathbb{Q}\mathcal{S}_r$-module, which is impossible since $\lambda\neq\mu$. $\square$

For any $K(\mathscr{H}_r\otimes\mathscr{H}_r)$ module M, let $p^1_{M_{\lambda-a_i}\otimes M_{\lambda-a_j}}: M\to M$ be the $K(\mathscr{H}_{r-1}\otimes\mathscr{H}_{r-1})$ projector with image the $M_{\lambda-a_i}\otimes M_{\lambda-a_j}$-isotypic component of M. For any $K\check{\mathscr{H}}_r$-module $\check{M}$ and $h\in\check{\mathscr{H}}_r$, let $m_h:\check{M}\to\check{M}$ denote right multiplication by h.

[4]The main change required is that (24) must be used in place of (22); these differ by some signs which end up being harmless.

Lemma 5.6. *Let $i, j \in [k_\lambda]$, $i \neq j$. There are the following equalities of $K\check{\mathscr{H}}_{r-1}$-module endomorphisms of $M_\lambda \otimes M_\lambda$.*

$$\begin{array}{llcl}
\text{(i)} & \check{p}^1_{\check{M}_{\lambda-a_i,\lambda-a_j}}\frac{1-\tau}{2} & = & \frac{1-\tau}{2}(p^1_{M_{\lambda-a_i}\otimes M_{\lambda-a_j}} + p^1_{M_{\lambda-a_j}\otimes M_{\lambda-a_i}}) \\
\text{(ii)} & \check{p}^1_{\check{M}_{\lambda-a_i,\lambda-a_j}}\check{p}^0_{S'\check{M}_\lambda} & = & \frac{1+\tau}{2}(p^1_{M_{\lambda-a_i}\otimes M_{\lambda-a_j}} + p^1_{M_{\lambda-a_j}\otimes M_{\lambda-a_i}}) \\
\text{(iii)} & \check{p}^1_{\check{M}_{\lambda-a_i,\lambda-a_j}}\check{p}^0_{S'\check{M}_\lambda}m_{\mathcal{P}_{r-1}}\check{p}^0_{S'\check{M}_\lambda} & = & \frac{1+\tau}{2}(p^1_{M_{\lambda-a_i}\otimes M_{\lambda-a_j}} + p^1_{M_{\lambda-a_j}\otimes M_{\lambda-a_i}})m_{\mathcal{P}_{r-1}}.
\end{array}$$

Proof. First note that for any $\mathscr{H}_r \otimes \mathscr{H}_r$-module M, there holds

$$\mathrm{Res}_{\check{\mathscr{H}}_{r-1}}\mathrm{Res}_{\mathscr{H}_{r-1}\otimes\mathscr{H}_{r-1}}M = \mathrm{Res}_{\check{\mathscr{H}}_{r-1}}M = \mathrm{Res}_{\check{\mathscr{H}}_{r-1}}\mathrm{Res}_{\check{\mathscr{H}}_r}M.$$

Statement (i) is immediate from the easy facts

$$\begin{aligned}
\check{p}^1_{\check{M}_{\lambda-a_i,\lambda-a_j}} &= p^1_{M_{\lambda-a_i}\otimes M_{\lambda-a_j}} + p^1_{M_{\lambda-a_j}\otimes M_{\lambda-a_i}}, \\
\tau p^1_{M_{\lambda-a_i}\otimes M_{\lambda-a_j}} &= p^1_{M_{\lambda-a_j}\otimes M_{\lambda-a_i}}\tau.
\end{aligned}$$

This also shows that (i) holds with $1+\tau$ in place of $1-\tau$. Then

$$\check{p}^1_{\check{M}_{\lambda-a_i,\lambda-a_j}}\frac{1+\tau}{2} = \check{p}^1_{\check{M}_{\lambda-a_i,\lambda-a_j}}\check{p}^0_{\check{S}^2\check{M}_\lambda} = \check{p}^1_{\check{M}_{\lambda-a_i,\lambda-a_j}}(\check{p}^0_{S'\check{M}_\lambda} + \check{p}^0_{K\check{\epsilon}_+}) = \check{p}^1_{\check{M}_{\lambda-a_i,\lambda-a_j}}\check{p}^0_{S'\check{M}_\lambda}$$

proves (ii). Statement (iii) is immediate from (ii) and the fact that $\check{p}^0_{S'\check{M}_\lambda}$ is a $K\check{\mathscr{H}}_r$-module homomorphism. □

We say that the modules in a list are *essentially distinct irreducibles* if the nonzero modules in this list are distinct irreducibles.

Proposition 5.7. *Maintain the setup of §5.1 and assume $\lambda \in \mathscr{P}'_r$. If $\check{\Lambda}^2\check{M}_{\lambda-a_i}$, $i \in [k_\lambda]$, and $\check{M}_{\lambda-a_i,\lambda-a_j}$, $i < j$, $i, j \in [k_\lambda]$, are essentially distinct irreducible $K\check{\mathscr{H}}_{r-1}$-modules, then $\check{\Lambda}^2\check{M}_\lambda$ is an irreducible $K\check{\mathscr{H}}_r$-module.*

Proof. We work with the basis $\Lambda^2\Gamma'_\lambda$ of $\check{\Lambda}^2\check{M}_\lambda$.

Let $i > 1$ and assume $\check{\Lambda}^2\check{M}_{\lambda-a_1}$ is nonzero. We show that $\check{\Lambda}^2\check{M}_{\lambda-a_1}$ glues to $\check{M}_{\lambda-a_i,\lambda-a_1}$ as follows: given the assumptions, we can choose $T, U \in \mathrm{SYT}(\lambda)$ so that

(1) $T_{a_1} = r$ and there is an edge $T \underset{r-1}{\leftrightsquigarrow} T'$ with $T'_{a_i} = r$.

(2) $U \neq T$ and $U_{a_1} = r$.

If $s_{r-1} \notin R(C'_U)$, then $C'_T \otimes C'_U\mathcal{P}_{r-1}$ is computed using the fourth case of (22). Lemma 5.6 (i) yields the first equality and the projection lemma yields the equivalence in the following

$$\begin{aligned}
\check{p}^1_{\check{M}_{\lambda-a_i,\lambda-a_1}}(C'_T \wedge C'_U\tfrac{\mathcal{P}_{r-1}}{[2]}) &= \tfrac{1-\tau}{2}(p^1_{M_{\lambda-a_i}\otimes M_{\lambda-a_1}} + p^1_{M_{\lambda-a_1}\otimes M_{\lambda-a_i}})(C'_T \otimes C'_U\tfrac{\mathcal{P}_{r-1}}{[2]}) \\
&\equiv -\sum_{\substack{s_{r-1}\in R(C'_A),\\ A_{a_i}=r}} \mu(A,T)(\tilde{C}'_A)^J \wedge (\tilde{C}'_U)^J - \sum_{\substack{s_{r-1}\in R(C'_B),\\ B_{a_i}=r}} \mu(B,U)(\tilde{C}'_T)^J \wedge (\tilde{C}'_B)^J \\
&= -\sum_{\substack{s_{r-1}\in R(C'_A),\\ A_{a_i}=r}} \mu(A,T)(\tilde{C}'_A)^J \wedge (\tilde{C}'_U)^J + \sum_{\substack{s_{r-1}\in R(C'_B),\\ B_{a_i}=r}} \mu(B,U)(\tilde{C}'_B)^J \wedge (\tilde{C}'_T)^J.
\end{aligned}$$

The equivalence is mod $u(\mathscr{L}_{\lambda-a_i} \otimes \mathscr{L}_{\lambda-a_1} \oplus \mathscr{L}_{\lambda-a_1} \otimes \mathscr{L}_{\lambda-a_i})$. The final line is nonzero because $(\tilde{C}'_{T'})^J \wedge (\tilde{C}'_U)^J$ appears in the left sum, $U \neq T$, and $\Gamma'_{\lambda-a_i} \otimes \Gamma'_{\lambda-a_1}$ is a K_0-basis of $\mathscr{L}_{\lambda-a_i} \otimes \mathscr{L}_{\lambda-a_1} \subseteq \check{M}_{\lambda-a_i,\lambda-a_1}$.[5] A similar (but easier) argument shows that $\check{p}^1_{\check{M}_{\lambda-a_i,\lambda-a_1}}(C'_T \wedge C'_U \frac{\mathcal{P}_{r-1}}{[2]})$ is nonzero in the case $s_{r-1} \in R(C'_U)$. Thus since $C'_T \wedge C'_U \in \check{\Lambda}^2 \check{M}_{\lambda-a_1}$ by Corollary 3.3, $\check{\Lambda}^2 \check{M}_{\lambda-a_1}$ glues to $\check{M}_{\lambda-a_i,\lambda-a_1}$ $(i > 1)$.

We next show that[6] $\check{M}_{\lambda-a_1,\lambda-a_2} \oplus \check{\Lambda}^2 \check{M}_{\lambda-a_1}$ glues to $\check{\Lambda}^2 \check{M}_{\lambda-a_2}$. Since we can assume $\check{\Lambda}^2 \check{M}_{\lambda-a_2}$ is nonzero, we can choose $T, U \in \mathrm{SYT}(\lambda)$ so that

(1) $T_{a_1} = r$ and there is an edge $T \underset{r-1}{\leftrightsquigarrow} T'$ with $T'_{a_2} = r$.
(2) $U_{a_2} = r$, and $U \neq T'$.

If $s_{r-1} \notin R(C'_U)$, then $C'_T \otimes C'_U \mathcal{P}_{r-1}$ is computed using the fourth case of (22). A careful application of the projection lemma shows that

$$\begin{aligned}\check{p}^1_{\check{\Lambda}^2 \check{M}_{\lambda-a_2}}(C'_T \wedge C'_U \tfrac{\mathcal{P}_{r-1}}{[2]}) &= \tfrac{1-\tau}{2} \check{p}^1_{M_{\lambda-a_2} \otimes M_{\lambda-a_2}}(C'_T \otimes C'_U \tfrac{\mathcal{P}_{r-1}}{[2]}) \\ &\equiv - \sum_{\substack{s_{r-1} \in R(C'_A), \\ A_{a_2} = r}} \mu(A, T) (\tilde{C}'_A)^J \wedge (\tilde{C}'_U)^J \mod u\Lambda^2 \mathscr{L}_{\lambda-a_2}\end{aligned}$$

The last line is nonzero because $(\tilde{C}'_{T'})^J \wedge (\tilde{C}'_U)^J$ appears in the sum, $U \neq T'$, and $\Lambda^2 \Gamma'_{\lambda-a_2}$ is a K_0-basis of $\Lambda^2 \mathscr{L}_{\lambda-a_2}$. A similar (but easier) argument shows that $\check{p}^1_{\check{\Lambda}^2 \check{M}_{\lambda-a_2}}(C'_T \wedge C'_U \frac{\mathcal{P}_{r-1}}{[2]})$ is nonzero in the case $s_{r-1} \in R(C'_U)$. Thus since $C'_T \wedge C'_U \in \check{M}_{\lambda-a_1,\lambda-a_2} \oplus \check{\Lambda}^2 \check{M}_{\lambda-a_1}$ by Corollary 3.3, $\check{M}_{\lambda-a_1,\lambda-a_2} \oplus \check{\Lambda}^2 \check{M}_{\lambda-a_1}$ glues to $\check{\Lambda}^2 \check{M}_{\lambda-a_2}$. Note that this argument still works if $\check{\Lambda}^2 \check{M}_{\lambda-a_1} = 0$.

Repeating the arguments of the previous two paragraphs, one shows that $\bigoplus_{1 < i \leq k_\lambda} \check{M}_{\lambda-a_1,\lambda-a_i} \oplus \bigoplus_{i \in \{1,2\}} \check{\Lambda}^2 \check{M}_{\lambda-a_i}$ glues to $\check{M}_{\lambda-a_2,\lambda-a_j}$ for $j > 2$, $\bigoplus_{\substack{1 \leq i < j \leq k_\lambda, \\ i \leq 2}} \check{M}_{\lambda-a_i,\lambda-a_j} \oplus \bigoplus_{i \in \{1,2\}} \check{\Lambda}^2 \check{M}_{\lambda-a_i}$ glues to $\check{\Lambda}^2 \check{M}_{\lambda-a_3}$, etc., which shows that all the irreducible constituents of $\mathrm{Res}_{K\check{\mathscr{H}}_{r-1}} \check{\Lambda}^2 \check{M}_\lambda$ are contained in a single $K\check{\mathscr{H}}_r$-irreducible. □

Proposition 5.8. *Maintain the setup of §5.1 and assume $\lambda \in \mathscr{P}'_r$. If $S'\check{M}_{\lambda-a_i}$, $i \in [k_\lambda]$ and $\check{M}_{\lambda-a_i,\lambda-a_j}$, $i < j$, $i, j \in [k_\lambda]$, are essentially distinct irreducible $K\check{\mathscr{H}}_{r-1}$-modules, then $S'\check{M}_\lambda$ is an irreducible $K\check{\mathscr{H}}_r$-module.*

Note that $S'\check{M}_\lambda$ does not necessarily have a multiplicity-free decomposition into $K\check{\mathscr{H}}_{r-1}$-irreducibles, but the proof method explained in §5.1 still gives most of the proof. The $K\check{\mathscr{H}}_{r-1}$-irreducible $K\check{\epsilon}_+$ may appear with multiplicity more than one, so it is handled separately.

Proof. We work with the basis $\Gamma_\lambda \otimes \Gamma'_\lambda$ of $M_\lambda \otimes M_\lambda$.

[5]Throughout this proof $\check{M}_{\lambda-a_i,\lambda-a_1}$ is understood as a $K\check{\mathscr{H}}_{r-1}$-submodule of $\mathrm{Res}_{K\check{\mathscr{H}}_{r-1}} \check{\Lambda}^2 \check{M}_\lambda$.

[6]By definition, $\check{M}_{\lambda-a_1,\lambda-a_2} = \check{M}_{\lambda-a_2,\lambda-a_1}$; we work with the former here to keep notation more consistent with other parts of the proof of Theorem 5.1.

If $k_\lambda = 1$, then $\mathrm{Res}_{K\mathscr{H}_{r-1}} S'\check{M}_\lambda \cong S'\check{M}_{\lambda-a_1}$, so the result holds. Assume $k_\lambda > 1$. First we show that $\check{M}_{\lambda-a_{k_\lambda},\lambda-a_1}$ glues to $\check{M}_{\lambda-a_i,\lambda-a_j}$ for $i > j$, $(i,j) \neq (k_\lambda, 1)$, as follows: choose $T, U \in \mathrm{SYT}(\lambda)$ so that

(1) $T_{a_{k_\lambda}} = r$, and if $i \neq k_\lambda$ then there is an edge $T \underset{r-1}{\leftrightsquigarrow} T'$ with $T'_{a_i} = r$.
(2) $U_{a_1} = r$, and if $j \neq 1$ then there is an edge $U \underset{r-1}{\leftrightsquigarrow} U'$ with $U'_{a_j} = r$.

Put $x = C_T \otimes C'_U$. We wish to show that

$$\check{p}^1_{\check{M}_{\lambda-a_i,\lambda-a_j}} \check{p}^0_{S'\check{M}_\lambda} m_{\mathcal{P}_{r-1}} \check{p}^0_{S'\check{M}_\lambda} x = \tfrac{1+\tau}{2}(p^1_{M_{\lambda-a_i}\otimes M_{\lambda-a_j}} + p^1_{M_{\lambda-a_j}\otimes M_{\lambda-a_i}})(x\mathcal{P}_{r-1}) \tag{30}$$

is nonzero (the equality is by Lemma 5.6 (iii)). This is shown in three cases.

The case $i \neq k_\lambda$ and $j \neq 1$: $s_{r-1} \notin R(C_T)$ and $s_{r-1} \notin R(C'_U)$, so $x\mathcal{P}_{r-1}$ is computed using the fourth case of (23). There holds

$$\begin{aligned}
&\tfrac{1+\tau}{2}(p^1_{M_{\lambda-a_i}\otimes M_{\lambda-a_j}} + p^1_{M_{\lambda-a_j}\otimes M_{\lambda-a_i}})(x\mathcal{P}_{r-1}) \\
&\equiv \sum_{\substack{s_{r-1}\in R(C_A),\\ s_{r-1}\in R(C'_B),\\ A_{a_i}=r,\ B_{a_j}=r}} \mu(A,T)\mu(B,U)(\tilde{C}_A)^J \cdot (\tilde{C}'_B)^J + \sum_{\substack{s_{r-1}\in R(C_A),\\ s_{r-1}\in R(C'_B),\\ A_{a_j}=r,\ B_{a_i}=r}} \mu(A,T)\mu(B,U)(\tilde{C}'_B)^J \cdot (\tilde{C}_A)^J, \\
&\equiv \sum_{\substack{s_{r-1}\in R(C_A),\\ s_{r-1}\in R(C'_B),\\ A_{a_i}=r,\ B_{a_j}=r}} \mu(A,T)\mu(B,U)(\tilde{C}'_A)^J \cdot (\tilde{C}'_B)^J + \sum_{\substack{s_{r-1}\in R(C_A),\\ s_{r-1}\in R(C'_B),\\ A_{a_j}=r,\ B_{a_i}=r}} \mu(A,T)\mu(B,U)(\tilde{C}'_B)^J \cdot (\tilde{C}'_A)^J,
\end{aligned}$$

where the first equivalence is by the projection lemma, the second is by Theorem 3.4, and the equivalences are mod $u\mathscr{L}_{\lambda-a_i} \otimes \mathscr{L}_{\lambda-a_j}$. The last line is nonzero because $(\tilde{C}'_{T'})^J \cdot (\tilde{C}'_{U'})^J$ appears in the left sum, the coefficients $\mu(A,T)\mu(B,U)$ are nonnegative (Theorem 3.1), and $\Gamma'_{\lambda-a_i} \otimes \Gamma'_{\lambda-a_j}$ is a K_0-basis of $\mathscr{L}_{\lambda-a_i} \otimes \mathscr{L}_{\lambda-a_j}$.

The case $i \neq k_\lambda, j = 1$ (the $i = k_\lambda, j \neq 1$ case is similar): $x\mathcal{P}_{r-1}$ is computed using the third or fourth case of (23). A careful application of the projection lemma yields

$$\begin{aligned}
&\tfrac{1+\tau}{2}(p^1_{M_{\lambda-a_i}\otimes M_{\lambda-a_j}} + p^1_{M_{\lambda-a_j}\otimes M_{\lambda-a_i}})(x\tfrac{\mathcal{P}_{r-1}}{[2]}) \\
&\equiv \pm \sum_{\substack{s_{r-1}\in R(C_A),\\ A_{a_i}=r}} \mu(A,T)(\tilde{C}_A)^J \cdot (\tilde{C}'_U)^J \mod u\mathscr{L}_{\lambda-a_i} \otimes \mathscr{L}_{\lambda-a_j}
\end{aligned}$$

The second line is nonzero because $(\tilde{C}_{T'})^J \cdot (\tilde{C}'_U)^J$ appears in sum and $\Gamma_{\lambda-a_i} \otimes \Gamma'_{\lambda-a_j}$ is a K_0-basis of $\mathscr{L}_{\lambda-a_i} \otimes \mathscr{L}_{\lambda-a_j}$.

It follows from Proposition 4.6 (ii) that $\check{p}^0_{S'\check{M}_\lambda} x = C_T \cdot C'_U$. Then by Lemma 5.6 (ii) and Corollary 3.3, $\check{p}^1_{\check{M}_{\lambda-a_{k_\lambda},\lambda-a_1}} \check{p}^0_{S'\check{M}_\lambda} x = C_T \cdot C'_U$, so $\check{p}^0_{S'\check{M}_\lambda} x \in \check{M}_{\lambda-a_{k_\lambda},\lambda-a_1} \subseteq S'\check{M}_\lambda$. Hence the left-hand side of (30) being nonzero implies that $\check{M}_{\lambda-a_{k_\lambda},\lambda-a_1}$ glues to $\check{M}_{\lambda-a_i,\lambda-a_j}$.

Fix $i \in [k_\lambda - 1]$ and set $\nu = \lambda - a_i$. Now we show that $\bigoplus_{j\leq i} \check{M}_{\lambda-a_{k_\lambda},\lambda-a_j}$ glues to $S'\check{M}_\nu$. Choose $T, U \in \mathrm{SYT}(\lambda)$ so that

(1) $T_{a_{k_\lambda}} = r$ and there is an edge $T \underset{r-1}{\leftrightsquigarrow} T'$ with $T'_{a_i} = r$.
(2) $U_{a_i} = r$ and $U \neq T'$.

This is possible since we can assume $S'\check{M}_\nu$ is nonzero, which is equivalent to $|\mathrm{SYT}(\nu)| > 1$. Then $C_T \cdot C'_U \mathcal{P}_{r-1}$ is computed using the third or fourth case of (23) with $\cdot$ in place of $\otimes$. A careful application of the projection lemma yields the first equivalence below

$$\begin{aligned}\check{p}^1_{S'\check{M}_\nu} p^1_{M_\nu \otimes M_\nu}\big(C_T \cdot C'_U \tfrac{\mathcal{P}_{r-1}}{[2]}\big) &\equiv \check{p}^1_{S'\check{M}_\nu}\Big(\pm \sum_{\substack{s_{r-1}\in R(C_A),\\ A_{a_i}=r}} \mu(A,T)(\tilde{C}_A)^J \cdot (\tilde{C}'_U)^J \Big) \\ &\equiv \pm \sum_{\substack{s_{r-1}\in R(C_A),\\ A_{a_i}=r}} \mu(A,T)\check{p}^1_{S'\check{M}_\nu}\big((\tilde{C}'_A)^J \cdot (\tilde{C}'_U)^J\big) \mod u\check{p}^0_{S'\check{M}_\nu}(\mathscr{L}_\nu \otimes \mathscr{L}_\nu).\end{aligned}$$

The second equivalence is by Theorem 3.4. It follows from Lemma 5.4 and $U \neq T'$ that the second line is nonzero. By an argument similar to that in the previous paragraph, $C_T \cdot C'_U = \check{p}^0_{S'\check{M}_\lambda}(C_T \otimes C'_U) \in \bigoplus_{j\leq i} \check{M}_{\lambda - a_{k_\lambda}, \lambda - a_j}$, hence $\bigoplus_{j\leq i} \check{M}_{\lambda - a_{k_\lambda}, \lambda - a_j}$ glues to $S'\check{M}_\nu$.

By an argument similar to the $i = 1$ case of the previous paragraph, $\check{M}_{\lambda - a_{k_\lambda}, \lambda - a_1}$ glues to $S'\check{M}_{\lambda - a_{k_\lambda}}$.

Let $X_\epsilon \subseteq S'\check{M}_\lambda$ be the isotypic component of $\mathrm{Res}_{K\check{\mathscr{H}}_{r-1}} S'\check{M}_\lambda$ of irreducible type $K\check{\epsilon}_+$ and let $X^{\mathbf{A}}_\epsilon := \bigcap_{i\in[r-2]} \ker(m_{\mathcal{Q}_i})$ be an integral form of X_ϵ, where $m_{\mathcal{Q}_i} : S'\check{M}^{\mathbf{A}}_\lambda \to S'\check{M}^{\mathbf{A}}_\lambda$ is right multiplication by $\mathcal{Q}_i$; there holds $K \otimes_{\mathbf{A}} X^{\mathbf{A}}_\epsilon \cong X_\epsilon$. To complete the proof, it suffices to show that $x\mathscr{H}_r \nsubseteq X^{\mathbf{A}}_\epsilon$ for any $x \in X^{\mathbf{A}}_\epsilon$. If $x\mathscr{H}_r \subseteq X^{\mathbf{A}}_\epsilon$, then $\mathrm{Res}_{\mathbb{Q}\mathcal{S}_{r-1}}(x\mathscr{H}_r|_{u=1})$ is a direct sum of copies of the trivial $\mathbb{Q}\mathcal{S}_{r-1}$-module (where $N|_{u=1}$ of an $\mathbf{A}$-module $N_{\mathbf{A}}$ is defined to be $\mathbb{Q} \otimes_{\mathbf{A}} N_{\mathbf{A}}$, the map $\mathbf{A} \to \mathbb{Q}$ given by $u \mapsto 1$). It follows that $x\mathscr{H}_r|_{u=1}$ is a direct sum of copies of the trivial $\mathbb{Q}\mathcal{S}_r$-module. But this is impossible since there are no copies of the trivial $\mathbb{Q}\mathcal{S}_r$-module in $S'\check{M}_\lambda|_{u=1}$. □

5.3. **Completing the proof.**

Proof of Theorem 5.1. The proof is by induction on r. Given that the $K\check{\mathscr{H}}_{r-1,2}$-irreducibles of the theorem are distinct, it follows from Propositions 5.3, 5.5, 5.7, and 5.8 that the $K\check{\mathscr{H}}_{r,2}$-modules in (1)–(4) are irreducible. The list of $K\check{\mathscr{H}}_{r,2}$-irreducibles is complete because $\mathrm{Res}_{K\check{\mathscr{H}}_{r,2}} K(\mathscr{H}_{r,2} \otimes \mathscr{H}_{r,2})$ is a faithful $K\check{\mathscr{H}}_{r,2}$-module and all the $K\check{\mathscr{H}}_{r,2}$-irreducible constituents of $\check{M}_{\lambda,\mu}$ appear in the list. Also, the split semisimplicity of $K\check{\mathscr{H}}_{r,2}$ follows from the proofs of Propositions 5.3, 5.5, 5.7, and 5.8 since these work just as well over any field extension of K. We now must show that the irreducibles in the list are distinct.

For this we apply Proposition 5.2 and refine the cases as follows:

(1a) $\mathrm{Res}_{K\check{\mathscr{H}}_{r-1,2}} \check{M}_{\lambda,\mu} \cong \bigoplus_{i\in[k_\lambda], j\in[k_\mu]} \check{M}_{\lambda-a_i,\mu-b_j}$, if $|\lambda \cap \mu| < r-1$.
(1b) $\mathrm{Res}_{K\check{\mathscr{H}}_{r-1,2}} \check{M}_{\lambda,\mu} \cong \bigoplus_{\substack{i\in[k_\lambda], j\in[k_\mu],\\ (i,j)\neq(k,l)}} \check{M}_{\lambda-a_i,\mu-b_j} \oplus S'\check{M}_\nu \oplus \check{\Lambda}^2\check{M}_\nu \oplus K\check{\epsilon}_+$, where $\nu = \lambda - a_k = \mu - b_l$ and $\nu \neq (r-1)$.
(1b′) $\mathrm{Res}_{K\check{\mathscr{H}}_{r-1,2}} \check{M}_{(r),(r-1,1)} \cong \check{M}_{(r-1),(r-2,1)} \oplus K\check{\epsilon}_+$.

(2) $\mathrm{Res}_{K\check{\mathscr{H}}_{r-1,2}} S'\check{M}_\lambda \cong \bigoplus_{1\leq i<j\leq k_\lambda} \check{M}_{\lambda-a_i,\lambda-a_j} \oplus \bigoplus_{i\in[k_\lambda]} S'\check{M}_{\lambda-a_i} \oplus K\check{\epsilon}_+^{\oplus k_\lambda -1}$, for $+\lambda \in \check{\mathscr{P}}_{r,2}$, $\lambda \neq (r-1,1)$.
(2′) $\mathrm{Res}_{K\check{\mathscr{H}}_{r-1,2}} S'\check{M}_{(r-1,1)} \cong \check{M}_{(r-1),(r-2,1)} \oplus S'\check{M}_{(r-2,1)} \oplus K\check{\epsilon}_+$, $r>2$.
(3) $\mathrm{Res}_{K\check{\mathscr{H}}_{r-1,2}} \check{\Lambda}^2\check{M}_\lambda \cong \bigoplus_{1\leq i<j\leq k_\lambda} \check{M}_{\lambda-a_i,\lambda-a_j} \oplus \bigoplus_{i\in[k_\lambda]} \check{\Lambda}^2\check{M}_{\lambda-a_i}$, for $-\lambda \in \check{\mathscr{P}}_{r,2}$, $\lambda \neq (r-1,1)$.
(3′) $\mathrm{Res}_{K\check{\mathscr{H}}_{r-1,2}} \check{\Lambda}^2\check{M}_{(r-1,1)} \cong \check{M}_{(r-1),(r-2,1)} \oplus \check{\Lambda}^2\check{M}_{(r-2,1)}$, $r>2$.
(4) $\mathrm{Res}_{K\check{\mathscr{H}}_{r-1,2}} K\check{\epsilon}_+ \cong K\check{\epsilon}_+$.

Note that for $r=3$, $S'\check{M}_{(1,1)}$ and $\check{\Lambda}^2\check{M}_{(1,1)}$ are zero in the right-hand sides of (2′) and (3′), respectively.

For $r\leq 3$, we check by hand that all these irreducibles are distinct. In particular, we must check that $\check{M}_{(3),(2,1)} \ncong S'\check{M}_{(2,1)}$, which happen to have isomorphic restrictions to $\check{\mathscr{H}}_{2,2}$.

Assuming that $r>3$ we will show that this list of irreducibles does not contain repetitions by showing that the irreducibles have distinct restrictions to $K\check{\mathscr{H}}_{r-1,2}$. We do this in two steps:

(A) The $K\check{\mathscr{H}}_{r-1,2}$ restrictions of any two irreducibles of a given type above are nonisomorphic.
(B) The $K\check{\mathscr{H}}_{r-1,2}$ restriction of an irreducible of type (α) is not isomorphic to the $K\check{\mathscr{H}}_{r-1,2}$ restriction of an irreducible of type (β), if $\alpha\neq\beta$.

Claim (A) is straightforward: for example, to see that two irreducibles of type (1a) are distinct, suppose M is a $K\check{\mathscr{H}}_{r,2}$-module of type (1a) and $\mathrm{Res}_{K\check{\mathscr{H}}_{r-1,2}} M \cong \bigoplus_{i\in[l]} \check{M}_{\nu^{(2i-1)},\nu^{(2i)}}$, for some $\nu^{(j)} \vdash r-1$. The set of partitions $\{\nu^{(i)}\cup\nu^{(j)} : i,j\in[2l],\ |\nu^{(i)}\cup\nu^{(j)}| = r\}$ consists of two partitions, call them λ and μ. Then $M = \check{M}_{\lambda,\mu}$.

For claim (B), we can look at which $K\check{\mathscr{H}}_{r-1,2}$ restrictions have no occurrences of an irreducible of the form $S'\check{M}_\nu$ and which ones have at least one occurrence of an irreducible of the form $S'\check{M}_\nu$, and similarly for the forms $\check{\Lambda}^2\check{M}_\nu$ and $K\check{\epsilon}_+$. This yields the claim (B) for all pairs of types except (2) and (2′), (3) and (3′), and (1b′) and (4) which are all easy to check directly.

The part of the theorem about the decomposition of $K(\mathscr{H}_{r,2}\otimes\mathscr{H}_{r,2})$-modules into $K\check{\mathscr{H}}_{r,2}$-irreducibles is immediate from the definitions in §4.4. □

Remark 5.9. It is possible that this proof would be easier using a Hecke algebra analog of Young's orthogonal basis (see [13]) instead of the lower and upper canonical bases. However, we believe it to be important to understand the action of $\check{\mathscr{H}}_{r,2}$ on the lower and upper canonical basis of $\check{M}_{\lambda,\mu}$ anyway. The canonical bases also have the advantage that all computations except the projections onto $K\check{\mathscr{H}}_{r-1,2}$-irreducible isotypic components take place over $\mathbf{A}[\frac{1}{[2]}]$ rather than K or some extension of K. Moreover, once the results of §3.6 are in place, the only thing we need to know about the $\mathcal{S}_r$-graphs Γ'_λ and Γ_λ are the edges corresponding to dual Knuth transformations; it is likely that a proof using a Hecke orthogonal basis would amount to showing the existence of certain dual Knuth transformations in a similar way.

6. Seminormal bases

We recall the definition of a seminormal basis from [12], observe that $K\check{\mathscr{H}}_{r,2}$-irreducibles have seminormal bases, and give combinatorial labels for the elements of these bases.

Definition 6.1. Given a chain of semisimple K-algebras $K \cong H_1 \subseteq H_2 \subseteq \cdots \subseteq H_r$ and an H_r-module N_λ, a *seminormal basis* of N_λ is a K-basis B of N_λ compatible with the restrictions in the following sense: there is a partition $B = B_{\mu^1} \sqcup \cdots \sqcup B_{\mu^k}$ such that $N_\lambda \cong N_{\mu^1} \oplus \cdots \oplus N_{\mu^k}$ as H_{r-1}-modules, where $N_{\mu^i} = KB_{\mu^i}$. Further, there is a partition of each B_{μ^i} that gives rise to a decomposition of N_{μ^i} into H_{r-2}-irreducibles, and so on, all the way down to H_1.

If the restriction of an H_i-irreducible to H_{i-1} is multiplicity-free for all i, then a seminormal basis of an H_r-irreducible is unique up to a diagonal transformation.

A consequence of Theorem 5.1 and Proposition 5.2 is that the restriction of a $K\check{\mathscr{H}}_{r,2}$-irreducible to $K\check{\mathscr{H}}_{r-1,2}$ is multiplicity-free. Thus each $K\check{\mathscr{H}}_{r,2}$-irreducible $\check{M}_\alpha$, $\alpha \in \check{\mathscr{P}}_{r,2}$, has a seminormal basis $\check{\mathrm{SN}}_\alpha$ that is unique up to a diagonal transformation. We adopt the convention to take the seminormal basis with respect to the chain $K\check{\mathscr{H}}_{J_1} \subseteq \cdots \subseteq K\check{\mathscr{H}}_{J_{r-1}} \subseteq K\check{\mathscr{H}}_{J_r}$, where $J_i = \{s_1, \ldots, s_{i-1}\}$ and $\check{\mathscr{H}}_L$ (for $L \subseteq S$) is the subalgebra of $\check{\mathscr{H}}_{r,2}$ generated by $\mathcal{P}_s$, $s \in L$.

For $\lambda, \mu \vdash r$ with $\ell(\lambda), \ell(\mu) \leq 2$, $M_\lambda \otimes M_\mu$ has a multiplicity-free decomposition into $\check{\mathscr{H}}_{r,2}$-modules (by Theorem 5.1). Thus we can also define a seminormal basis $\check{\mathrm{SN}}_{\lambda,\mu}$ of $M_\lambda \otimes M_\mu$ to be the union of the seminormal bases of its $K\check{\mathscr{H}}_{r,2}$-irreducible constituents.

We are interested in these seminormal bases primarily as a tool for constructing a canonical basis of a $K\check{\mathscr{H}}_{r,2}$-irreducible that is compatible with its decomposition into irreducibles at $u = 1$, as described in [2, §19]. Even though the irreducibles of $K\check{\mathscr{H}}_{r,2}$ are close to those of $K(\mathscr{H}_{r,2} \otimes \mathscr{H}_{r,2})$, the seminormal basis $\check{\mathrm{SN}}_{\lambda,\mu}$ of $M_\lambda \otimes M_\mu$ using the chain $K\check{\mathscr{H}}_{J_1} \subseteq \cdots \subseteq K\check{\mathscr{H}}_{J_{r-1}} \subseteq K\check{\mathscr{H}}_{J_r}$ is significantly different from the seminormal basis using the chain $K(\mathscr{H}_{1,2} \otimes \mathscr{H}_{1,2}) \subseteq \cdots \subseteq K(\mathscr{H}_{r-1,2} \otimes \mathscr{H}_{r-1,2}) \subseteq K(\mathscr{H}_{r,2} \otimes \mathscr{H}_{r,2})$. Thus even though the representation theory of the nonstandard Hecke algebra alone is not enough to understand Kronecker coefficients, there is hope that the seminormal bases $\check{\mathrm{SN}}_{\lambda,\mu}$ will yield a better understanding of Kronecker coefficients.

Remark 6.2. The $K\check{\mathscr{H}}_6$-module $\check{M}_{(4,1,1),(3,2,1)}$ is irreducible and its $K\check{\mathscr{H}}_5$ restriction is not multiplicity-free. However, we suspect that $K\check{\mathscr{H}}_{r-1}$ restrictions of $K\check{\mathscr{H}}_r$-irreducibles are very often multiplicity-free and, if not, the multiplicities are small.

6.1. Combinatorics of seminormal bases.

For $\lambda, \mu \vdash r$, $\ell(\lambda), \ell(\mu) \leq 2$, define a bijection

$$\mathrm{SYT}(\lambda) \times \mathrm{SYT}(\mu) \xrightarrow{\alpha_{\lambda,\mu}} \check{\mathrm{SN}}_{\lambda,\mu}$$

inductively as follows. Maintain the notation of (1) for the outer corners of λ and μ. In what follows let $(T, U) \in \mathrm{SYT}(\lambda) \times \mathrm{SYT}(\mu)$ and i and j be such that $T_{a_i} = r$ and $U_{b_j} = r$. Let Y_λ be the tableau with entries $2c-1, 2c$ in column c for each column of λ of height 2. For convenience, we identify the basis $\check{\mathrm{SN}}_{\lambda,\mu}$ with the corresponding subset of one-dimensional subspaces of $M_\lambda \otimes M_\mu$.

(i) If $\lambda \neq \mu$, set

$$\alpha_{\lambda,\mu}(T,U) = \alpha_{\lambda-a_i,\mu-a_j}(T_{\lambda-a_i},\ U_{\lambda-a_j}).$$

(ii) If $\lambda = \mu$, then set

$$\alpha_{\lambda,\mu}(T,U) = \begin{cases} K\check{\epsilon}_+ \subseteq S^2 M_\lambda & \text{if } (T,U) = (Y_\lambda, Y_\lambda), \\ \alpha_{\lambda-a_i,\mu-a_j}(T_{\lambda-a_i},\ U_{\lambda-a_j}) & \text{otherwise,} \end{cases}$$

where $\alpha_{\lambda-a_i,\mu-a_j}(T_{\lambda-a_i},\ U_{\lambda-a_j})$ is interpreted as a seminormal basis element of

$$\begin{cases} M_{\lambda-a_i} \otimes M_{\lambda-a_j} \subseteq \operatorname{Res}_{K\check{\mathscr{H}}_{r-1,2}} M_\lambda \otimes M_\lambda & \text{if } i = j, \\ M_{\lambda-a_i} \otimes M_{\lambda-a_j} \subseteq \operatorname{Res}_{K\check{\mathscr{H}}_{r-1,2}} S'\check{M}_\lambda & \text{if } i < j, \\ M_{\lambda-a_i} \otimes M_{\lambda-a_j} \subseteq \operatorname{Res}_{K\check{\mathscr{H}}_{r-1,2}} \check{\Lambda}^2 \check{M}_\lambda & \text{if } i > j. \end{cases}$$

Given Proposition 5.2 and Theorem 5.1, it is clear that $\alpha_{\lambda,\mu}$ is a well-defined bijection.

Example 6.3. The seminormal basis element $\alpha_{(3,2),(3,2)}\left(\begin{matrix}1&2&4\\3&5\end{matrix}, \begin{matrix}1&3&4\\2&5\end{matrix}\right)$ is a nonzero element of $S'\check{M}_{(3,2)} \cap S'\check{M}_{(3,1)} \cap S'\check{M}_{(2,1)} \cap \check{M}_{(2),(1,1)}$, where these are modules for $K\check{\mathscr{H}}_{J_5}$, $K\check{\mathscr{H}}_{J_4}$, $K\check{\mathscr{H}}_{J_3}$, and $K\check{\mathscr{H}}_{J_2}$, respectively.

The next two tables partially describe the bijection $\alpha_{(3,2),(3,2)}$; they give the $K\check{\mathscr{H}}_{J_5}$ and $K\check{\mathscr{H}}_{J_4}$-irreducibles that contain the seminormal basis element corresponding to each $(T,U) \in \mathrm{SYT}((3,2)) \times \mathrm{SYT}((3,2))$, where row labels correspond to T and column labels correspond to U; the basis element just described is in bold.

	$\begin{matrix}1&2&3\\4&5\end{matrix}$	$\begin{matrix}1&2&4\\3&5\end{matrix}$	$\begin{matrix}1&3&4\\2&5\end{matrix}$	$\begin{matrix}1&2&5\\3&4\end{matrix}$	$\begin{matrix}1&3&5\\2&4\end{matrix}$
$\begin{matrix}1&2&3\\4&5\end{matrix}$	$S'\check{M}_{(3,2)}$	$S'\check{M}_{(3,2)}$	$S'\check{M}_{(3,2)}$	$S'\check{M}_{(3,2)}$	$S'\check{M}_{(3,2)}$
$\begin{matrix}1&2&4\\3&5\end{matrix}$	$\check{\Lambda}^2\check{M}_{(3,2)}$	$S'\check{M}_{(3,2)}$	$\mathbf{S'\check{M}_{(3,2)}}$	$S'\check{M}_{(3,2)}$	$S'\check{M}_{(3,2)}$
$\begin{matrix}1&3&4\\2&5\end{matrix}$	$\check{\Lambda}^2\check{M}_{(3,2)}$	$\check{\Lambda}^2\check{M}_{(3,2)}$	$S'\check{M}_{(3,2)}$	$S'\check{M}_{(3,2)}$	$S'\check{M}_{(3,2)}$
$\begin{matrix}1&2&5\\3&4\end{matrix}$	$\check{\Lambda}^2\check{M}_{(3,2)}$	$\check{\Lambda}^2\check{M}_{(3,2)}$	$\check{\Lambda}^2\check{M}_{(3,2)}$	$S'\check{M}_{(3,2)}$	$S'\check{M}_{(3,2)}$
$\begin{matrix}1&3&5\\2&4\end{matrix}$	$\check{\Lambda}^2\check{M}_{(3,2)}$	$\check{\Lambda}^2\check{M}_{(3,2)}$	$\check{\Lambda}^2\check{M}_{(3,2)}$	$\check{\Lambda}^2\check{M}_{(3,2)}$	$K\check{\epsilon}_+$

	$\begin{smallmatrix}1&2&3\\4&5\end{smallmatrix}$	$\begin{smallmatrix}1&2&4\\3&5\end{smallmatrix}$	$\begin{smallmatrix}1&3&4\\2&5\end{smallmatrix}$	$\begin{smallmatrix}1&2&5\\3&4\end{smallmatrix}$	$\begin{smallmatrix}1&3&5\\2&4\end{smallmatrix}$
$\begin{smallmatrix}1&2&3\\4&5\end{smallmatrix}$	$S'\check{M}_{(3,1)}$	$S'\check{M}_{(3,1)}$	$S'\check{M}_{(3,1)}$	$\check{M}_{(3,1),(2,2)}$	$\check{M}_{(3,1),(2,2)}$
$\begin{smallmatrix}1&2&4\\3&5\end{smallmatrix}$	$\check{\Lambda}^2\check{M}_{(3,1)}$	$S'\check{M}_{(3,1)}$	$\mathbf{S'\check{M}_{(3,1)}}$	$\check{M}_{(3,1),(2,2)}$	$\check{M}_{(3,1),(2,2)}$
$\begin{smallmatrix}1&3&4\\2&5\end{smallmatrix}$	$\check{\Lambda}^2\check{M}_{(3,1)}$	$\check{\Lambda}^2\check{M}_{(3,1)}$	$K\check{\epsilon}_+$	$\check{M}_{(3,1),(2,2)}$	$\check{M}_{(3,1),(2,2)}$
$\begin{smallmatrix}1&2&5\\3&4\end{smallmatrix}$	$\check{M}_{(2,2),(3,1)}$	$\check{M}_{(2,2),(3,1)}$	$\check{M}_{(2,2),(3,1)}$	$S'\check{M}_{(2,2)}$	$S'\check{M}_{(2,2)}$
$\begin{smallmatrix}1&3&5\\2&4\end{smallmatrix}$	$\check{M}_{(2,2),(3,1)}$	$\check{M}_{(2,2),(3,1)}$	$\check{M}_{(2,2),(3,1)}$	$\check{\Lambda}^2\check{M}_{(2,2)}$	$K\check{\epsilon}_+$

7. Enumerative consequence

Let $C_r = \frac{1}{r+1}\binom{2r}{r}$ be the r-th Catalan number. Theorem 5.1 has the following corollary.

Corollary 7.1. *The algebra $K\check{\mathscr{H}}_{r,2}$ has dimension $\binom{C_r}{2} - \binom{r}{\lfloor \frac{r}{2}\rfloor} + \lfloor \frac{r}{2}\rfloor + 2$.*

Proof. It is well known that $\dim_K(K\mathscr{H}_{r,2}) = C_r$ and therefore $\dim_K(KS^2\mathscr{H}_{r,2}) = \binom{C_r+1}{2}$. On the other hand, the list of irreducibles of $KS^2\mathscr{H}_{r,2}$ given in Proposition-Definition 4.2 and the split semisimplicity of $KS^2\mathscr{H}_{r,2}$ imply that

$$\dim_K(KS^2\mathscr{H}_{r,2}) = \sum_{\substack{\lambda\rhd\mu,\\ \ell(\lambda),\ell(\mu)\le 2}} (f_\lambda f_\mu)^2 + \sum_{\ell(\lambda)\le 2}\left(\binom{f_\lambda+1}{2}^2 + \binom{f_\lambda}{2}^2\right), \tag{31}$$

where $f_\lambda = \dim_K(M_\lambda) = |\mathrm{SYT}(\lambda)|$.

The list of $K\check{\mathscr{H}}_{r,2}$-irreducibles from Theorem 5.1 and the split semisimplicity of $K\check{\mathscr{H}}_{r,2}$ imply that

$$\dim_K(K\check{\mathscr{H}}_{r,2}) = \sum_{\substack{\lambda\rhd\mu,\\ \ell(\lambda),\ell(\mu)\le 2}} (f_\lambda f_\mu)^2 + \sum_{\ell(\lambda)\le 2,\ \lambda\ne(r)}\left(\left(\binom{f_\lambda+1}{2} - 1\right)^2 + \binom{f_\lambda}{2}^2\right) + 1. \tag{32}$$

Taking the difference of the right-hand sides of (31) and (32) then yields the first of the following string of equalities.

$$\begin{aligned}
\dim_K(K\check{\mathscr{H}}_{r,2}) &= \dim_K(KS^2\mathscr{H}_{r,2}) + \textstyle\sum_{\ell(\lambda)\le 2,\ \lambda\ne(r)}\left(-2\binom{f_\lambda+1}{2} + 1\right)\\
&= \binom{C_r+1}{2} - \textstyle\sum_{\ell(\lambda)\le 2} f_\lambda^2 - \sum_{\ell(\lambda)\le 2} f_\lambda + \left(\sum_{\ell(\lambda)\le 2} 1\right) + 1\\
&= \binom{C_r+1}{2} - C_r - \textstyle\sum_{\ell(\lambda)\le 2} f_\lambda + (\lfloor \frac{r}{2}\rfloor + 1) + 1\\
&= \binom{C_r}{2} - \binom{r}{\lfloor \frac{r}{2}\rfloor} + \lfloor \frac{r}{2}\rfloor + 2.
\end{aligned}$$

The second equality follows from $\dim_K(KS^2\mathscr{H}_{r,2}) = \binom{C_r+1}{2}$, the third equality comes from counting the dimension of the split semisimple algebra $K\mathscr{H}_{r,2}$ in two ways, and the

fourth from the fact that $\dim_{\mathbb{C}}(\mathrm{Ind}_P^{\mathcal{S}_r}\mathrm{triv}) = \sum_{\ell(\lambda)\leq 2} f_\lambda$, where P is the maximal parabolic subgroup $(\mathcal{S}_r)_{S\setminus s_{\lfloor \frac{r}{2} \rfloor}}$ of $\mathcal{S}_r$. $\square$

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